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Ferromagnetic Resonance
in Magnetic Tunnel Junctions
under High DC Biases

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Ferromagnetic Resonance in Magnetic Tunnel Junctions under High DC Biases

by

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Abstract

Ferromagnetic Resonance in Magnetic Tunnel Junctions under High DC Biases

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Ferromagnetic resonance (FMR) is a standard spectroscopic technique which is used to probe the magnetodynamics of ferromagnetic materials in order to understand and improve performance of spintronics applications such as magnetic random-access memory (MRAM). In our experiments, we use rf microwave currents to excite FMR in magnetic tunnel junctions (MTJs) via spin-transfer torque (STT-FMR) that allows us to electrically detect magnetodynamics by means of a small rectified voltage which develops across the MTJ at resonance. The MTJ pillars used in this work have diameters on the order of 100 nm and consist of free and pinned CoFeB layers separated by a MgO barrier with typical tunneling magnetoresistances (TMRs) of about 100% at room temperature. As expected, the frequency-field relationship of the observed resonances can be well fitted by Kittel's equation. However, as a function of the dc bias applied to the MTJ we observe an unexpected shift of the resonance field. This shift is symmetric about zero bias and may be a result of the out-of-plane voltage controlled magnetic anisotropy

(VCMA) in the otherwise in-plane magnetized MTJ. In addition to the effective field due to VCMA, an out-of-plane field was produced by canting the applied field. A generalized angular dependent version of Kittel's equation revealed little influence of the out-of-plane applied field with respect to the effective VCMA field. Also, two measurement techniques for detecting FMR, rf amplitude modulation and applied magnetic field modulation are reviewed and compared.

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Chapter 1: Introduction

For the past half century, charge based electronics using complementary metal oxide semiconductor (CMOS) technology has dominated computing architecture elements. Advancement in sync with Moore's Law has been challenging, but achievable. Recently however, CMOS technology has encountered obstacles in scalability due to high power density leading to heat and power consumption issues. As a result, significant effort is currently being committed to the development of alternative computer logic and memory devices. As a prospective replacement for CMOS transistors, magnetic tunnel junctions (MTJs) are promising for future computing applications requiring low power consumption and low performance such as the Internet of Things.[1]

A magnetic tunnel junction is a nanoscale multilayer device in which the basic construction consists of one insulating layer between two magnetic layers.[2]The relative orientation between the magnetization of each layer provides a way to encode information. If the relative orientation is parallel (antiparallel), the MTJ is in a low (high) resistance state each representing a binary digital value of zero or one. The insulating layer between the two magnetic layers separates the magnetic monodomains as well as establishes the tunneling magnetoresistance ratio (TMR) (the relative change in resistance between the parallel and antiparallel states.) A nonmagnetic metal layer can substitute the insulating layer, which denotes a spin valve. In such a case, the relative change in resistance is defined as giant magnetoresistance (GMR), the topic of the 2007 Nobel Prize in physics. The discovery of GMR in 1988 by Albert Fert and Peter Grünberg augmented research in non-volatile magnetic electronic devices triggering the field later called spintronics.[3]

Another noteworthy advancement was the discovery of spin-transfer torque (STT).[4–9]Previously, magnetization orientation was switched by localized Oersted fields created by an array of metallic lines which suffered from high power consumption and limited scalability.[10,11] In contrast, the STT effect permits the switching of the magnetization direction using a spin-polarized current[12,13] which enables a much simpler design, denser layout, and improved scalability down to 10 nm.[14] Auspiciously, magnetic random access memory (MRAM) using STT has the potential to build a universal memory, simultaneously bridging the gaps of speed, density, and non-volatility of static random-access memory (SRAM), dynamic random-access memory (DRAM), and flash memory, respectively.[15] However, the write current densities still exceed the values required for the widespread adoption of MRAM.

In this work, we investigate the behavior of in-plane magnetized MTJs subjected to a constant current bias. We focus on the bias current effects on ferromagnetic resonance (FMR) and inferring magnetic anisotropy induced by a bias voltage. FMR is a useful tool for investigating magnetodynamics which are important to fully characterize the switching process. A review of the theory of FMR is provided. We also compare two measurement techniques that can be used to measure FMR and discuss the benefits and drawbacks of using each.

Chapter 2: Theory of Ferromagnetic Resonance

2.1 INTRODUCTION

Ferromagnetic resonance (FMR) is a powerful technique which is used for determining many fundamental magnetic and magneto-dynamic properties that reveal significant nanoscale behavior of ferromagnetic materials and devices.† These properties are significant from both fundamental physics and applied perspectives.

The original discovery of ferromagnetic resonance was made unwittingly by V. K. Arkad'yev[25] when he observed the absorption of UHF (centimeter scale) radiation by ferromagnetic materials in 1911. In 1923, Ya. G. Dorfman[26] suggested that FMR was connected to the optical transitions from Zeeman splitting. In 1935 Landau and Lifshitz predicted ferromagnetic resonance at the Larmor frequency.[27] J. H. E. Griffiths and E. K. Zavoiskij independently verified Landau and Lifshitz in 1946 by demonstrating FMR[28–30] in thin films. Thin films are advantageous for study due to their likely single-domain structure and their consequent single-domain rotation magnetization process.[32]

In his experiment Griffiths' found the unusual result that the resonance frequencies he observed were greater than the calculated Larmor frequencies for electron spin by factors of about two to six. He attempted unsuccessfully to explain the anomaly by the introduction of the Lorentz cavity force. It was shown by Kittel that it is important to consider the dynamic coupling caused by the demagnetization field normal to the surface of the specimen.[33]This correction becomes significant in thin films as those used in the original experiment and modern experiments using MTJs since the demagnetization field is derived from Kittel's shape factors. The flatness of the thin films

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†Crystalline anisotropy energy constants, spectroscopic splitting Landé g-factors[16], magneto-mechanical (or gyromagnetic ratios)[17], surface anisotropy constants[18], the magnitude of the resonant static field and its dependence on field orientation, on sample thickness, and on temperature[19], magnetic damping constants[20],exchange coupling constants[21], magnetization saturation[22], coercive force[23], symmetry axes, spin torque contribution,etc.[24]

introduces crystallographic anisotropy such that there are two unique directions: in-plane and out-of-plane.[34]

FMR in Isotropic Materials

For an isotropic magnetic material in a constant magnetic field the magnetization precesses about the external field in equilibrium analogously to Electron Spin Resonance and Nuclear Magnetic Resonance. A perturbing perpendicular radio frequency (rf) magnetic field applied to the specimen can excite resonance of the magnetization in the system. If the perturbing transverse magnetic field is a rf field with frequency ω, then the magnetization will precess at its resonant frequency $\omega_0 = \gamma\vec{H}_0$ if $\omega \approx \omega_0$.[35,36]

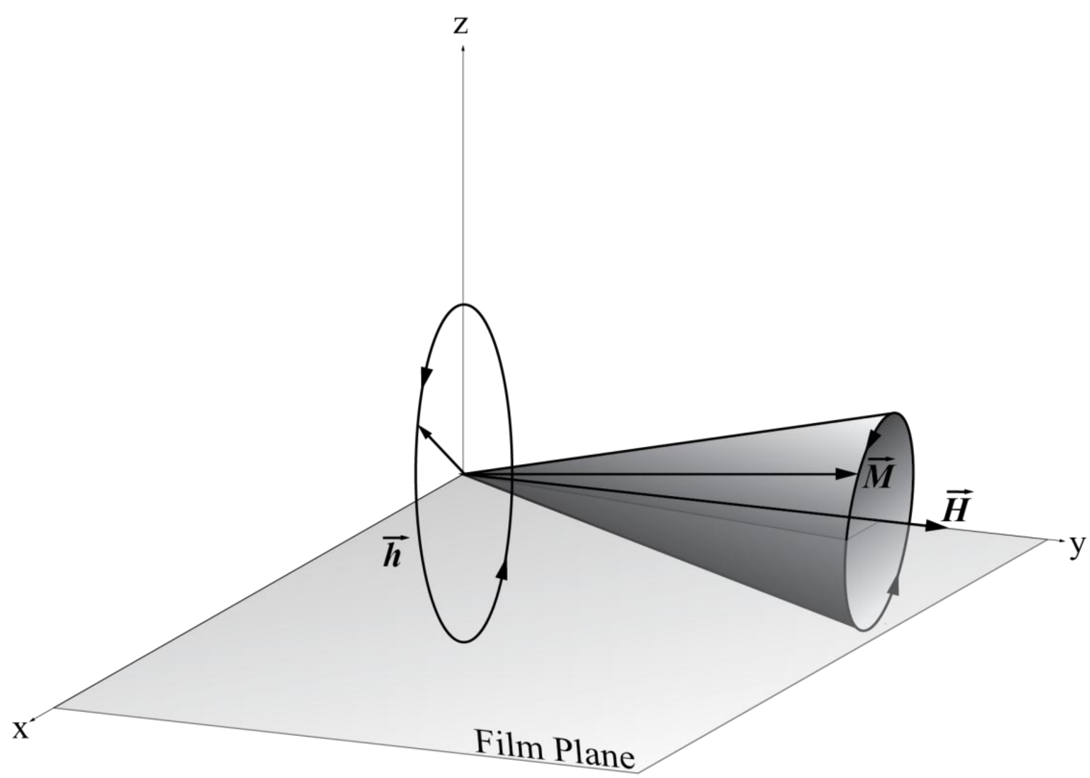

Figure 1: Precession of the magnetization vector $\vec{\boldsymbol{M}}$ in the static magnetic field $\vec{\boldsymbol{H}}_{\mathbf{0}}$ and a rf magnetic field $\vec{\boldsymbol{h}}$. Film plane is shown only for reference.

2.2 BASIC DERIVATION

Vonsovskii Coordinate System

The following derivation expounds upon Vonsovskii's book on ferromagnetic resonance.[35] Figure 2 shows the most used coordinate system described in Vonsovskii's work. The orientation of the static magnetic field $\vec{\boldsymbol{H}}_{\mathbf{0}}$ is defined by the azimuth and polar angles $\boldsymbol{\phi_H}$, $\boldsymbol{\theta_H}$ respectively. Likewise, the magnetization $\vec{\boldsymbol{M}}$ is defined by the azimuth and polar angles $\boldsymbol{\phi_M}$, $\boldsymbol{\theta_M}$ and the magnitude of the magnetization $\boldsymbol{M}$, which we assume is constant at the value of the saturation magnetization.

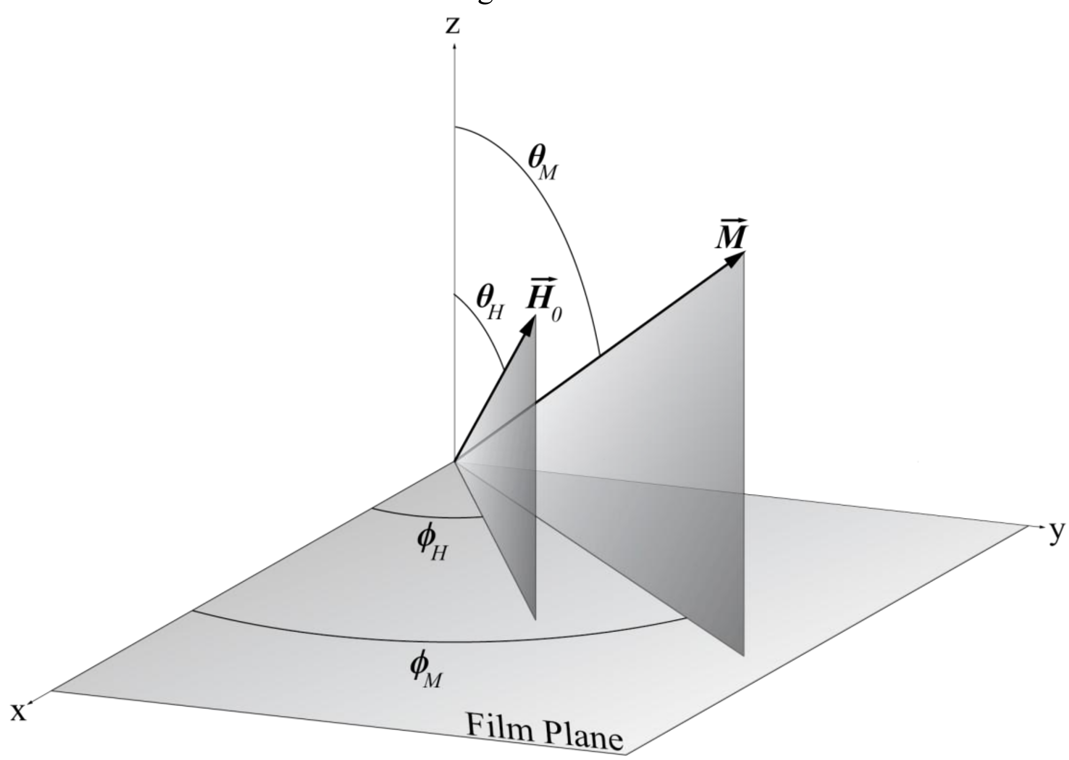

Figure 2: Variables describing the orientation of the static field $\vec{\boldsymbol{H}}_{\mathbf{0}}$, and magnetization vector $\vec{\boldsymbol{M}}$ in Vonsovskii spherical coordinates.

Effective Internal Field $\vec{H}$ Derived from Helmholtz Free Energy

Landau and Lifshitz discovered that the various interactions including shape and uniaxial anisotropy can be taken into account by assuming that the magnetization vector precesses about an internal effective field $\vec{H}$.[27] This effective field is derived from the Helmholtz free energy.

The effective magnetic field $\vec{H}$ acts on the magnetization and is defined by the three components H_M, H_{ϕ_M}, H_{θ_M} in spherical coordinates each of which correspond to the respective magnetization unit vector directions. The effective field is comprised of the influences from the static field, shape anisotropy, and voltage controlled anisotropy modeled here using uniaxial anisotropy.

We can express the internal effective field $\vec{H}$ using the Helmholtz free energy of the system.[37]

$$dF = -SdT - pdV + \vec{H} \cdot d\vec{M}$$

"In a state of thermodynamic equilibrium the direction of the magnetization vector M in a ferromagnet coincides with the direction of the effective field H_M, whose magnitude can be found using the [Helmholtz] free energy F."[38]

$$\left(\frac{\partial F}{\partial M}\right)_{T,V} = -H_M$$

Generalizing the expression for the effective field $\vec{H}$ to vector notation and expressing the equilibrium orientation,

$$\vec{H} = -\vec{\nabla}_{\vec{M}} F = -\left(\frac{\partial F}{\partial x}\widehat{M}_x + \frac{\partial F}{\partial y}\widehat{M}_y + \frac{\partial F}{\partial z}\widehat{M}_z\right)$$

This equation can be expressed in spherical coordinates,

$$\vec{\mathrm{H}} = -\vec{\nabla}_{\vec{\mathrm{M}}}\mathrm{F} = -\left(\frac{\partial \mathrm{F}}{\partial \mathrm{M}_r}\widehat{\mathrm{M}} + \frac{1}{\mathrm{M}\sin\theta_\mathrm{M}}\frac{\partial \mathrm{F}}{\partial \phi_\mathrm{M}}\widehat{\phi}_\mathrm{M} + \frac{1}{\mathrm{M}}\frac{\partial \mathrm{F}}{\partial \theta_\mathrm{M}}\widehat{\theta}_\mathrm{M}\right)$$

$$\mathrm{H}_\mathrm{M} = -\left(\frac{\partial \mathrm{F}}{\partial \mathrm{M}}\right) = -\mathrm{F}_\mathrm{M}$$

$$\mathrm{H}_{\phi_\mathrm{M}} = -\frac{1}{\mathrm{M}\sin\theta_\mathrm{M}}\left(\frac{\partial \mathrm{F}}{\partial \phi_\mathrm{M}}\right) = -\frac{1}{\mathrm{M}\sin\theta_\mathrm{M}}\mathrm{F}_{\phi_\mathrm{M}}$$

$$\mathrm{H}_{\theta_\mathrm{M}} = -\frac{1}{\mathrm{M}}\left(\frac{\partial \mathrm{F}}{\partial \theta_\mathrm{M}}\right) = -\frac{1}{\mathrm{M}}\mathrm{F}_{\theta_\mathrm{M}}$$

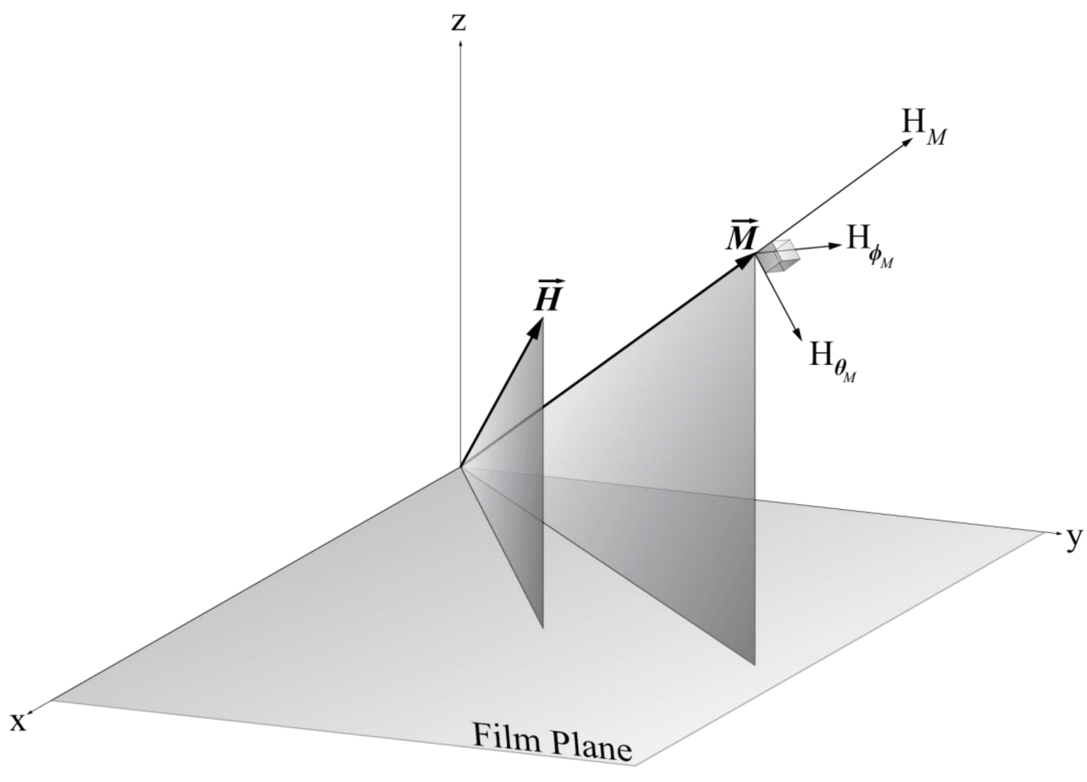

Figure 3: continued next page.

Figure 3: Components of the effective field $\boldsymbol{H_M}$, $\boldsymbol{H_{\phi_M}}$, $\boldsymbol{H_{\theta_M}}$ in spherical coordinates which act on the magnetization causing precession.

The direction of $\vec{H}$ is the same as $\vec{M}$ in equilibrium therefore,

$$\left(\frac{\partial F}{\partial \phi_M}\right) = F_{\phi_M} = 0 \qquad \left(\frac{\partial F}{\partial \theta_M}\right) = F_{\theta_M} = 0$$

with equilibrium angles $\phi_{M,0}, \theta_{M,0}$.

Equations of Motion

For a homogeneously magnetized specimen, the magnetization can be represented as a magnetic top whose movement is described by Pauli's equation which reduces to the vector equation,[39]

$$\dot{\vec{M}} = \gamma \vec{T}$$

where $\vec{T}$ is the torque acting on the magnetization $\vec{M}$, $[\vec{M} \times \vec{H}]$, and γ is the gyromagnetic ratio.

$$\dot{\vec{M}} = \gamma [\vec{M} \times \vec{H}]$$

Resonant Frequency in Rectangular Coordinates

Smit and Beljers[40] pioneered a general way to solve for the resonant frequency from the Helmholtz free energy. Suhl[41] and Tannenwald[42] also independently developed the same method around the same time. Smit and Beljers let the equilibrium direction of the magnetization vector be the ζ-direction. For small angles of deviation from equilibrium the magnetization vector can be described by two perpendicular directions ξ and η.

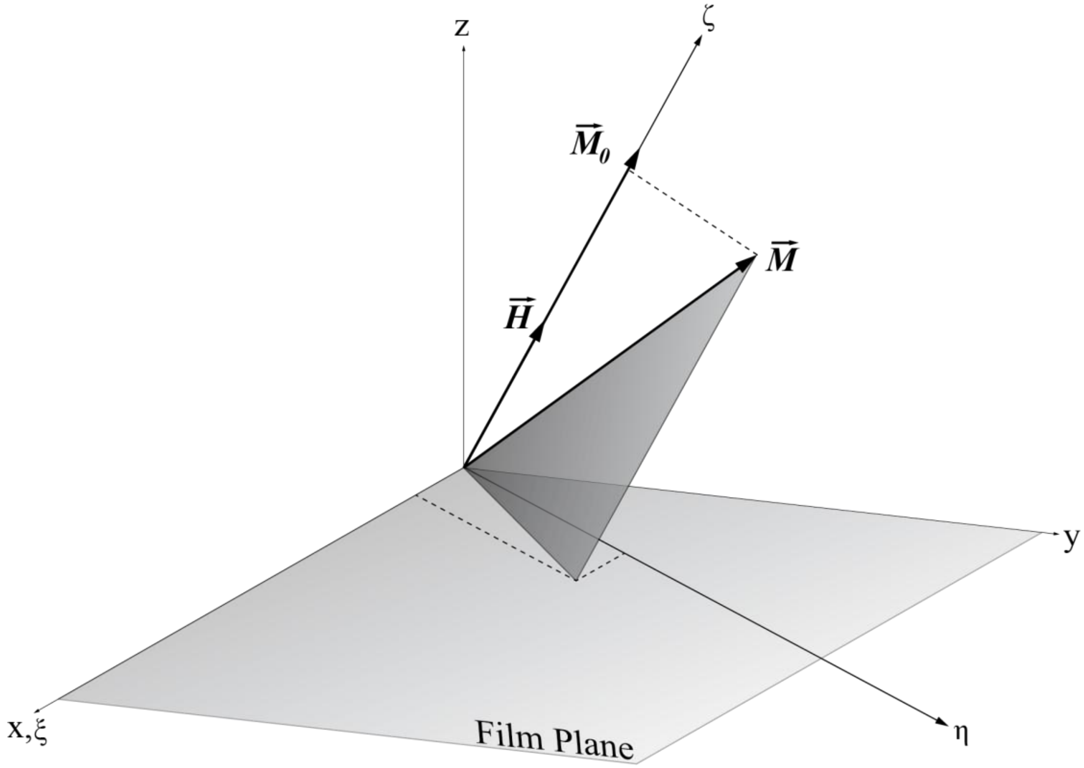

Figure 4: Smit and Beljers rectangular coordinate system transformation.

Applying the equations of motion and the expression for the effective field we get,

$$-M\dot{\eta} = \gamma \ \partial F/\partial \xi$$

$$M\dot{\xi} = \gamma \ \partial F/\partial \eta$$

For small deviations from the equilibrium position we may use the first terms of the Taylor series for F.

$$F = F_0 + \frac{1}{2}(F_{\xi\xi}\xi^2 + 2F_{\xi\eta}\xi\eta + F_{\eta\eta}\eta^2)$$

Plugging this into the equations of motion,

$$-M\dot{\eta} = \gamma\left(F_{\xi\xi}\xi + F_{\xi\eta}\eta\right)$$

$$M\dot{\xi} = \gamma\left(F_{\eta\xi}\xi + F_{\eta\eta}\eta\right)$$

If this system of differential equations has solutions which vary harmonically in time the resonance frequency is expressed as follows,

$$\left(\frac{\omega_0}{\gamma}\right)^2 = \frac{1}{M^2}\left[F_{\xi\xi}F_{\eta\eta} - F_{\xi\eta}^2\right]$$

Resonant Frequency in Spherical Coordinates

Smit and Beljers' rectangular coordinate derivation is theoretically simple, yet experimentally we control the angles of orientation of the static field. Therefore it is more convenient for us to derive an expression for the resonant frequency in spherical coordinates. We start by defining the relevant vectors.

Magnetization Vector $\overrightarrow{M}$

$$M_x = M\cos\phi_M \sin\theta_M \qquad M_y = M\sin\phi_M \sin\theta_M \qquad M_z = M\cos\theta_M$$

Time Derivative of Magnetization Vector $\dot{\overrightarrow{M}}$

$$\dot{M}_x = M\left(-\dot{\phi}_M \sin\phi_M \sin\theta_M + \dot{\theta}_M \cos\phi_M \cos\theta_M\right)$$

$$\dot{M}_y = M\left(\dot{\phi}_M \cos\phi_M \sin\theta_M + \dot{\theta}_M \sin\phi_M \cos\theta_M\right)$$

$$\dot{M}_z = -M\,\dot{\theta}_M \sin\theta_M$$

Effective Field Vector $\vec{H}$ in Spherical Coordinates

$$H_M = H_x \cos\phi_M \sin\theta_M + H_y \sin\phi_M \sin\theta_M + H_z \cos\theta_M$$

$$H_{\phi_M} = H_x \cos\phi_M \cos\theta_M + H_y \sin\phi_M \cos\theta_M - H_z \sin\theta_M$$

$$H_{\theta_M} = -H_x \sin\phi_M + H_y \cos\phi_M$$

In order to use the equations of motion, we need every vector in Cartesian coordinates since the equations of motion take the form of a cross product. (In spherical coordinates unit vectors are changing as the components of the unit vectors change.)

Using the coordinate system and intuition (the ϕ_M component acts only in the x-y plane):

$$H_z = H_M \cos\theta_M - H_{\theta_M} \sin\theta_M$$

With H_z determined we can solve for the two remaining components H_x, H_y with the two equations H_M, H_{ϕ_M}. We can start by solving for H_x,

$$H_x = \frac{-H_{\phi_M} + H_y \cos\phi_M}{\sin\phi_M}$$

Plugging H_x into H_M,

$$H_M = \left(\frac{-H_{\phi_M}}{\sin\phi_M} + \frac{H_y \cos\phi_M}{\sin\phi_M}\right)\cos\phi_M \sin\theta_M + H_y \sin\phi_M \sin\theta_M + H_z \cos\theta_M$$

$$H_M = H_y\left(\frac{\cos^2\phi_M}{\sin\phi_M}\cos\phi_M \sin\theta_M + \sin\phi_M \sin\theta_M\right) - H_{\phi_M}\frac{\cos\phi_M \sin\theta_M}{\sin\phi_M}$$
$$+ \left(H_M \cos\theta_M - H_{\theta_M} \sin\theta_M\right)\cos\theta_M$$

$$H_y = \frac{H_M - H_M \cos^2\theta_M + H_{\phi_M}\frac{\cos\phi_M \sin\theta_M}{\sin\phi_M} + H_{\theta_M}\cos\theta_M \sin\theta_M}{\frac{\cos^2\phi_M}{\sin\phi_M}\sin\theta_M + \sin\phi_M \sin\theta_M}$$

$$H_y = \frac{H_M \sin^2\theta_M + H_{\phi_M}\frac{\cos\phi_M \sin\theta_M}{\sin\phi_M} + H_{\theta_M}\cos\theta_M \sin\theta_M}{\sin\theta_M\left[\frac{\cos^2\phi_M}{\sin\phi_M} + \sin\phi_M\right]}$$

$$H_y = \frac{H_M \sin\theta_M + H_{\phi_M}\frac{\cos\phi_M}{\sin\phi_M} + H_{\theta_M}\cos\theta_M}{\frac{1}{\sin\phi_M}\left[\cos^2\phi_M + \sin^2\phi_M\right]}$$

$$H_y = H_M \sin\phi_M \sin\theta_M + H_{\phi_M}\cos\phi_M + H_{\theta_M}\sin\phi_M \cos\theta_M$$

Now that we have H_y in purely spherical components, we can back substitute H_y into the expression for H_x.

$$H_x = \frac{-H_{\phi_M} + H_y \cos\phi_M}{\sin\phi_M}$$

$$H_x = \frac{-H_{\phi_M} + \left(H_M \sin\phi_M \sin\theta_M + H_{\phi_M}\cos\phi_M + H_{\theta_M}\sin\phi_M \cos\theta_M\right)\cos\phi_M}{\sin\phi_M}$$

$$H_x = \frac{-H_{\phi_M}}{\sin\phi_M} + \left(H_M \frac{\sin\phi_M}{\sin\phi_M}\cos\phi_M \sin\theta_M + H_{\phi_M}\frac{\cos^2\phi_M}{\sin\phi_M} + H_{\theta_M}\frac{\sin\phi_M}{\sin\phi_M}\cos\phi_M \cos\theta_M\right)$$

$$H_x = H_M \cos\phi_M \sin\theta_M + H_{\phi_M}\frac{1}{\sin\phi_M}\left(\cos^2\phi_M - 1\right) + H_{\theta_M}\cos\phi_M \cos\theta_M$$

Which leads to the Cartesian form,

Effective Field Vector $\vec{H}$ in Cartesian Coordinates

$$H_x = H_M \cos\phi_M \sin\theta_M - H_{\phi_M}\sin\phi_M + H_{\theta_M}\cos\phi_M \cos\theta_M$$

$$H_y = H_M \sin\phi_M \sin\theta_M + H_{\phi_M} \cos\phi_M + H_{\theta_M} \sin\phi_M \cos\theta_M$$

$$H_z = H_M \cos\theta_M - H_{\theta_M} \sin\theta_M$$

We then plug these results into the equations of motion.

For the x direction,

$$\dot{M}_x = \gamma\left[M_y H_z - M_z H_y\right]$$

$$\dot{M}_x = \gamma\left[M \sin\phi_M \sin\theta_M \left(H_M \cos\theta_M - H_{\theta_M} \sin\theta_M\right)\right.$$
$$\left. - M \cos\theta_M \left(H_M \sin\phi_M \sin\theta_M + H_{\phi_M} \cos\phi_M + H_{\theta_M} \sin\phi_M \cos\theta_M\right)\right]$$

$$\dot{M}_x = \gamma M\left[H_M (\sin\phi_M \cos\theta_M \sin\theta_M - \sin\phi_M \cos\theta_M \sin\theta_M)\right.$$
$$\left. - H_{\phi_M} \cos\phi_M \cos\theta_M + H_{\theta_M} (-\sin\phi_M \sin^2\theta_M - \sin\phi_M \cos^2\theta_M)\right]$$

$$\dot{M}_x = \gamma M\left(-H_{\phi_M} \cos\phi_M \cos\theta_M - H_{\theta_M} \sin\phi_M\right)$$

For the y direction,

$$\dot{M}_y = -\gamma[M_z H_x - M_x H_z]$$

$$\dot{M}_y = -\gamma\left[M \cos\theta_M \left(H_M \cos\phi_M \sin\theta_M - H_{\phi_M} \sin\phi_M + H_{\theta_M} \cos\phi_M \cos\theta_M\right)\right.$$
$$\left. + M \cos\phi_M \sin\theta_M \left(H_M \cos\theta_M - H_{\theta_M} \sin\theta_M\right)\right]$$

$$\dot{M}_y = -\gamma M\left[(\cos\phi_M \cos\theta_M \sin\theta_M - \cos\phi_M \cos\theta_M \sin\theta_M) H_M + H_{\phi_M} \sin\phi_M \cos\theta_M\right.$$
$$\left. - H_{\theta_M} (\cos\phi_M \cos^2\theta_M + \cos\phi_M \sin^2\theta_M)\right]$$

$$\dot{M}_y = \gamma M\left(H_{\phi_M} \sin\phi_M \cos\theta_M - H_{\theta_M} \cos\phi_M\right)$$

For the z direction,

$$\dot{M}_z = \gamma\left[M_x H_y - M_y H_x\right]$$

$$\dot{M}_z = \gamma\Big[M\cos\phi_M \sin\theta_M \left(H_M \sin\phi_M \sin\theta_M + H_{\phi_M} \cos\phi_M + H_{\theta_M} \sin\phi_M \cos\theta_M\right)$$
$$- M\sin\phi_M \sin\theta_M \left(H_M \cos\phi_M \sin\theta_M - H_{\phi_M} \sin\phi_M\right.$$
$$\left. + H_{\theta_M} \cos\phi_M \cos\theta_M\right)\Big]$$

$$\dot{M}_z = \gamma M\big[H_M (\cos\phi_M \sin\phi_M \sin^2\theta_M - \cos\phi_M \sin\phi_M \sin^2\theta_M)$$
$$+ H_{\phi_M} (\cos^2\phi_M \sin\theta_M + \sin^2\phi_M \sin\theta_M)$$
$$+ H_{\theta_M} (\cos\phi_M \sin\phi_M \cos\theta_M \sin\theta_M - \cos\phi_M \sin\phi_M \cos\theta_M \sin\theta_M)\big]$$

$$\dot{M}_z = \gamma M \sin\theta_M H_{\phi_M}$$

Resulting in $\dot{\vec{M}}$,

$$\dot{M}_x = \gamma M\left(-H_{\phi_M} \cos\phi_M \cos\theta_M - H_{\theta_M} \sin\phi_M\right)$$

$$\dot{M}_y = \gamma M\left(H_{\phi_M} \sin\phi_M \cos\theta_M - H_{\theta_M} \cos\phi_M\right)$$

$$\dot{M}_z = \gamma M \sin\theta_M H_{\phi_M}$$

We now apply the time derivative of the magnetization vector to the equations of motion.

$$\dot{M}_x = M\left(-\dot{\phi}_M \sin\phi_M \sin\theta_M + \dot{\theta}_M \cos\phi_M \cos\theta_M\right)$$

$$= \gamma M\left(-H_{\phi_M} \cos\phi_M \cos\theta_M - H_{\theta_M} \sin\phi_M\right)$$

$$\dot{M}_y = M\left(\dot{\phi}_M \cos\phi_M \sin\theta_M + \dot{\theta}_M \sin\phi_M \cos\theta_M\right)$$

$$= \gamma M\left(H_{\phi_M} \sin\phi_M \cos\theta_M - H_{\theta_M} \cos\phi_M\right)$$

$$\dot{M}_z = -M\,\dot{\theta}_M \sin\theta_M = \gamma M \sin\theta_M\, H_{\phi_M}$$

Immediately we can express the equation from the z component,

$$\dot{\theta}_M = \gamma H_{\phi_M}$$

Plugging this result into the equation for the y component yields

$$\left(\dot{\phi}_M \cos\phi_M \sin\theta_M + \gamma H_{\phi_M} \sin\phi_M \cos\theta_M\right) = \gamma\left(H_{\phi_M} \sin\phi_M \cos\theta_M - H_{\theta_M} \cos\phi_M\right)$$

$$\dot{\phi}_M \sin\theta_M = -\gamma H_{\theta_M}$$

This results in the two equations of motion

$$\dot{\theta}_M = \gamma H_{\phi_M}$$

$$\dot{\phi}_M \sin\theta_M = -\gamma H_{\theta_M}$$

Solving System of Linear Equations for Resonant Frequency

We now have two equations with two unknowns ϕ_M, θ_M, which we would normally be able to solve given initial conditions. Unfortunately, we are unable to solve exactly for the unknowns because the differential equations are nonlinear. However, if we make a linearized approximation assuming small deviations from equilibrium we can solve for the linearized variables using matrix diagonalization.

Linearized Helmholtz Free Energy

Let us define $\Delta\phi_M, \Delta\theta_M$ such that

$$\Delta\phi_M(t) = \phi_M(t) + \phi_{M,0}$$

$$\Delta\theta_M(t) = \theta_M(t) + \theta_{M,0}$$

Assuming small values of $\Delta\phi_M$, $\Delta\theta_M$ allows us to consider only the first order terms in the Taylor Series expansion of the Helmholtz Free Energy.

$$F_{\phi_M} = F_{\phi_M\phi_M}\Delta\phi_M + F_{\phi_M\theta_M}\Delta\theta_M\Big|_{\phi_{M,0},\theta_{M,0}} F_\vartheta = F_{\phi_M\theta_M}\Delta\phi_M + F_{\theta_M\theta_M}\Delta\theta_M\Big|_{\phi_{M,0},\theta_{M,0}}$$

We can combine the equations of motion and the results from the linearized Helmholtz free energy to construct a system of linear equations.

Plugging in the expression for Helmholtz free energy into the equation for the effective field $\vec{H}$,

$$H_{\phi_M} = \frac{-1}{M\sin\theta_{M,0}}\left[F_{\phi_M\phi_M}\Delta\phi_M + F_{\phi_M\theta_M}\Delta\theta_M\right]H_{\theta_M}$$

$$= -\frac{1}{M}\left[F_{\phi_M\theta_M}\Delta\phi_M + F_{\theta_M\theta_M}\Delta\theta_M\right]$$

Inserting the result into the Equations of Motion

$$\dot{\theta}_M = -\gamma\frac{1}{M\sin\theta_{M,0}}\left[F_{\phi_M\phi_M}\Delta\phi_M + F_{\phi_M\theta_M}\Delta\theta_M\right]$$

$$\dot{\phi}_M = -\gamma\frac{1}{M\sin\theta_{M,0}}\left[F_{\phi_M\theta_M}\Delta\phi_M + F_{\theta_M\theta_M}\Delta\theta_M\right]$$

Dropping the Δ, in matrix form, $\overleftrightarrow{A}\vec{x} = \vec{b}$

$$\frac{\gamma}{M\sin\theta_{M,0}}\begin{bmatrix} -F_{\phi_M\theta_M} & -F_{\theta_M\theta_M} \\ F_{\phi_M\phi_M} & F_{\phi_M\theta_M} \end{bmatrix}\begin{bmatrix} \phi_M \\ \theta_M \end{bmatrix} = \begin{bmatrix} \dot{\phi}_M \\ \dot{\theta}_M \end{bmatrix}$$

This system of two linear equations becomes an eigenvalue problem if the angular functions are periodic.

$$\theta_M \sim e^{i\omega t}\phi_M \sim e^{i\omega t}$$

$$\dot{\theta_M} \sim i\omega\theta_M\dot{\phi_M} \sim i\omega\phi_M$$

Resulting in, $\overleftrightarrow{A}\vec{x} = \lambda\vec{x}$

$$\begin{bmatrix} -F_{\phi_M\theta_M} & -F_{\theta_M\theta_M} \\ F_{\phi_M\phi_M} & F_{\phi_M\theta_M} \end{bmatrix}\begin{bmatrix}\phi_M \\ \theta_M\end{bmatrix} = i\omega\frac{M\sin\theta_{M,0}}{\gamma}\begin{bmatrix}\phi_M \\ \theta_M\end{bmatrix}$$

This takes the eigenvalue form, $(\overleftrightarrow{A} - \overleftrightarrow{I}\lambda)\vec{x} = \vec{0}$

$$\begin{bmatrix} -F_{\phi_M\theta_M} - i\omega\frac{M\sin\theta_{M,0}}{\gamma} & -F_{\theta_M\theta_M} \\ F_{\phi_M\phi_M} & F_{\phi_M\theta_M} - i\omega\frac{M\sin\theta_{M,0}}{\gamma} \end{bmatrix}\begin{bmatrix}\phi_M \\ \theta_M\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}.$$

Which only has nontrivial solutions if $\det\left(\overleftrightarrow{A} - \overleftrightarrow{I}\lambda\right) = 0$.

$$\begin{vmatrix} -F_{\phi_M\theta_M} - i\omega_0\frac{M\sin\theta_{M,0}}{\gamma} & -F_{\theta_M\theta_M} \\ F_{\phi_M\phi_M} & F_{\phi_M\theta_M} - i\omega_0\frac{M\sin\theta_{M,0}}{\gamma} \end{vmatrix}$$

$$= \left(-F_{\phi_M\theta_M} - i\omega_0\frac{M\sin\theta_{M,0}}{\gamma}\right)\left(F_{\phi_M\theta_M} - i\omega_0\frac{M\sin\theta_{M,0}}{\gamma}\right) + F_{\phi_M\phi_M}F_{\theta_M\theta_M} = 0$$

$$F_{\phi_M\theta_M}^2 + {\omega_0}^2\frac{M^2\sin^2\theta_{M,0}}{\gamma^2} - F_{\phi_M\phi_M}F_{\theta_M\theta_M} = 0$$

$$\omega_0 = \frac{\gamma}{M\sin\theta_{M,0}}\sqrt{F_{\phi_M\phi_M}F_{\theta_M\theta_M} - F_{\phi_M\theta_M}^2}$$

Or in a form more familiar to most in the literature,

$$\left(\frac{\omega_0}{\gamma}\right)^2 = \frac{1}{M^2 \sin^2 \theta_{M,0}} \left[F_{\phi_M \phi_M} F_{\theta_M \theta_M} - F^2_{\phi_M \theta_M}\right]$$

However, this solution does not always hold for $\sin \theta_{M,0} = 0$.

2.3 Specific Form of Helmholtz Free Energy

As mentioned before, we include the static field contribution to the free energy, which results in the Zeeman energy term, the shape anisotropy energy term, and the voltage controlled magnetic anisotropy energy term modeled as uniaxial anisotropy.

$$F = F_{Zeeman} + F_{Shape} + F_{Uniaxial}$$

According to the work by Akulov, the expression for the uniaxial anisotropy term is comprised of an exponential series with respect to the directional cosines, α, of the magnetization vector relative to the crystal's principal axes. In the case for crystals with uniaxial symmetry we have[43]

$$F_{Uniaxial} = K_0' + K_1'\alpha^2 + K_2'\alpha^4 + \cdots = K_0 + K_1\beta^2 + K_2\beta^4 + \cdots$$

Where β is the sine of the angle between the magnetization and the axis of symmetry. In our case,

$$\beta = \sin \theta_M$$

Here we have approximated the series out to the second order term in β,

$$F = K_1 \sin^2 \theta_M - \vec{M} \cdot \vec{H_0} + \int \vec{M} \cdot \overleftrightarrow{N}\vec{M}\, dv$$

$$F = K_1 \sin^2 \theta_M$$
$$- MH[\cos\phi_H \sin\theta_H \cos\phi_M \sin\theta_M \; + \sin\phi_H \sin\theta_H \sin\phi_M \sin\theta_M$$
$$+ \cos\theta_H \cos\theta_M] \; + \frac{1}{2}M^2[N_x \cos^2\phi_M \sin^2\theta_M + N_y \sin^2\phi_M \sin^2\theta_M$$
$$+ N_z \cos^2\theta_M]$$

Where ϕ_H and θ_H represent the azimuth and polar angles respectively of the external field. $\overleftrightarrow{N}$ is the shape anisotropy tensor and N_x, N_y, N_z are the shape factors.[44] As in Vonsovskii's work, we will fully define the constant external field direction as in-plane$\phi_H = {}^{\pi}/_{2}$, $\theta_H = {}^{\pi}/_{2}$, due to mathematical complexity. By defining the external field we set the direction of magnetization at equilibrium $\phi_{M,0} = {}^{\pi}/_{2}$, $\theta_{M,0} = {}^{\pi}/_{2}$. In order to reflect our experimental setup we will assign shape factors corresponding to a thin film in CGS units. $N_x = 0,\ N_y = 0,\ N_z = 4\pi$. CGS units are used here to depict agreement with literature. In SI units the equivalent shape factors are $N_x = 0,\ N_y = 0,\ N_z = 1,$ however, using SI units we must be aware of when it is appropriate to multiply by μ_0.

$$F = K\sin^2\theta_M - MH\sin\phi_M \sin\theta_M + \frac{1}{2}M^2\, 4\pi\cos^2\theta_M$$

Differentiating,

$$F_{\theta_M} = 2K\sin\theta_M\cos\theta_M - MH\sin\phi_M\cos\theta_M - \frac{4\pi}{2}M^2\, 2\cos\theta_M\sin\theta_M$$

$$F_{\phi_M} = -MH\cos\phi_M\sin\theta_M.$$

And again,

$$F_{\theta_M\theta_M} = 2K(\cos^2\theta_M - \sin^2\theta_M) + MH\sin\phi_M\sin\theta_M - 4\pi M^2(\cos\theta_M - \sin^2\theta_M)$$

$$F_{\phi_M \phi_M} = MH \sin\phi_M \sin\theta_M \; F_{\phi_M \theta_M} = MH \cos\phi_M \cos\theta_M = F_{\theta_M \phi_M}$$

Evaluating the second derivatives at the equilibrium point $\phi_M = \phi_{M,0}, \theta_M = \theta_{M,0}$,

$$F_{\theta_M \theta_M} = 2K\left(\cos^2\theta_{M,0} - \sin^2\theta_{M,0}\right) + MH \sin\phi_{M,0} \sin\theta_{M,0}$$
$$- 4\pi M^2\left(\cos\theta_{M,0} - \sin^2\theta_{M,0}\right)$$

$$F_{\phi_M \phi_M} = MH \sin\phi_{M,0} \sin\theta_{M,0} \; F_{\varphi\theta_M} = MH \cos\phi_{M,0} \cos\theta_{M,0} = F_{\theta_M \phi_M}$$

Simplifying with the imposed conditions,

$$F_{\theta_M \theta_M} = -2K + MH + 4\pi M^2$$

$$F_{\phi_M \phi_M} = MHF_{\phi_M \theta_M} = 0 = F_{\theta_M \phi_M}.$$

Plugging this result into the resonant frequency equation,

$$\omega_0 = \frac{\gamma}{M}\sqrt{F_{\phi_M \phi_M} F_{\theta_M \theta_M} - F^2_{\phi_M \theta_M}}$$

$$\omega_0 = \frac{\gamma}{M}\sqrt{MH(MH - 2K + 4\pi M^2)}$$

$$\omega_0 = \gamma\sqrt{H(H - \frac{2K}{M} + 4\pi M)}$$

In the case of zero uniaxial anisotropy $K = 0$, the result agrees with Kittel's result[45]

$$\omega_0 = \gamma\sqrt{H(H + 4\pi M)}$$

2.4 CHAPPERT DERIVATION OF RESONANT FREQUENCY

Chappert[46], who worked in CGS units, introduced a new coordinate system to the FMR frequency derivation most likely to simplify the analysis and avoid the limitation associated with the Smit Beljers (SB) method[40] in which $\sin\theta_{M,0} = 0$. One other justification for using Chappert's analysis originates from the author's inability to produce Okada's results from the SB method. Okada et. al. have published work on the effect of VCMA on perpendicular anisotropy MTJs, a topic of study very close to this work. Producing Okada's results, which are in the same coordinate system as the SB method, is possible operating in Chappert's coordinate system with the appropriate coordinate transformations. This process is shown in appendices B through D.

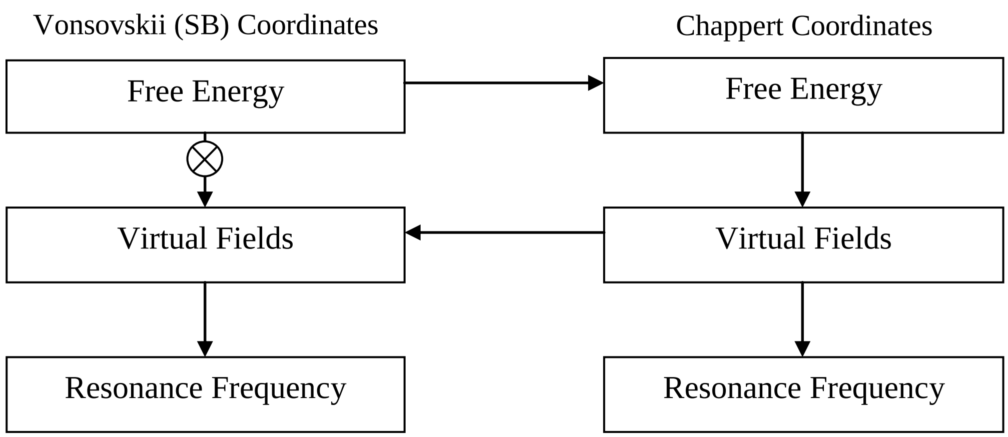

Figure 5: Diagram of coordinate systems transformed through in order to produce Okada's FMR result.

Figure 6 shows this coordinate system with ϕ'_M and θ'_M describing the magnetization orientation, and likewise ϕ'_H and θ'_H describing the static field orientation.

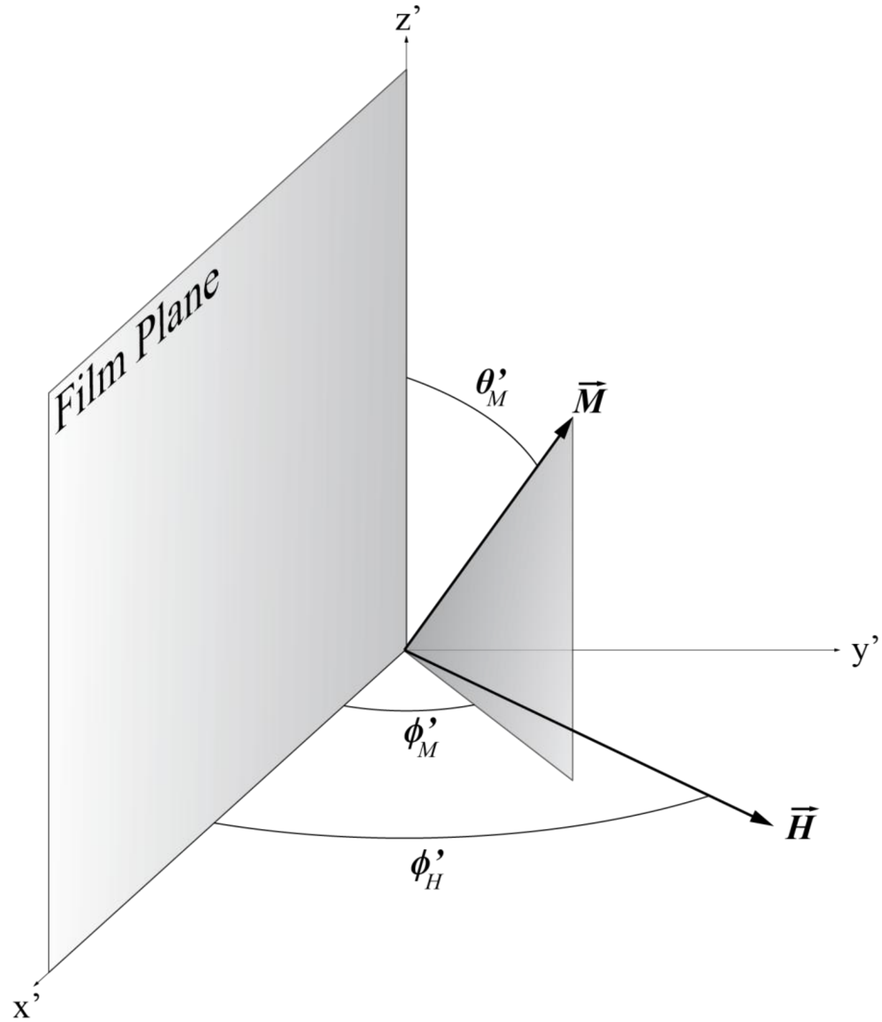

Figure 6: Coordinate system used by Chappert in his derivation of the FMR frequency.

The conversion equations between the Vonsovskii and Chappert coordinates are as follows,

Magnetization Orientation Transformation

$$\cos\theta_M = \sin\theta'_M \sin\phi'_M \qquad \cos\phi'_M = \frac{\sin\phi_M \sin\theta_M}{\sqrt{1-\cos^2\phi_M \sin^2\theta_M}}$$

$$\sin\theta_M = \sqrt{1-\sin^2\theta'_M \sin^2\phi'_M} \qquad \sin\phi'_M = \frac{\cos\theta_M}{\sqrt{1-\cos^2\phi_M \sin^2\theta_M}}$$

$$\cos\phi_M = \frac{\cos\theta'_M}{\sqrt{1-\sin^2\theta'_M \sin^2\phi'_M}} \qquad \cos\theta'_M = \cos\phi_M \sin\theta_M$$

$$\sin\phi_M = \frac{\sin\theta'_M \cos\phi'_M}{\sqrt{1-\sin^2\theta'_M \sin^2\phi'_M}} \qquad \sin\theta'_M = \sqrt{1-\cos^2\phi_M \sin^2\theta_M}$$

Static Field Orientation Transformations

Due to azimuthal symmetry reflected in the Vonsovskii coordinate system, Chappert imposed an arbitrary azimuthal angle regarding the orientation of the static field, $\theta'_H = \frac{\pi}{2}$. Therefore the only meaningful coordinate describing the static field is ϕ'_H. We arrive at the following transformations given these conditions.

$$\cos\theta_H = \sin\phi'_H \qquad \cos\phi'_H = \sin\theta_H$$

$$\sin\theta_H = \cos\phi'_H \qquad \sin\phi'_H = \cos\theta_H$$

$$\cos\phi_H = \frac{\cos\theta'_H}{\cos\phi'_H} \qquad \cos\theta'_H = 0$$

$$\sin\phi_H = 1 \qquad \sin\theta'_H = 1$$

These transformations can be seen in detail in Appendix A

The energy density function appropriate to Chappert's coordinate system is given by the expression

$$F = -HM\cos(\phi'_H - \phi'_M)\sin\theta'_M + \frac{1}{2}(4\pi M^2)\sin^2\theta'_M\sin^2\phi'_M$$
$$- (K_1 + 2K_2)\sin^2\theta'_M\sin^2\phi'_M + K_2\sin^4\theta'_M\sin^4\phi'_M$$

where the first term represents the Zeeman energy, the second the magnetostatic energy, and the last two terms the uniaxial anisotropy energy with the easy axis parallel to the y' axis. Here the uniaxial anisotropy energy is estimated out to the second nonconstant term in the series which is fourth order in β.

The expression of the free energy in Chappert's work is equivalent to the expression in Vonsovskii[35] coordinates as seen in Appendix B

$$F = -HM(\cos(\phi_H - \phi_M)\sin\theta_H\sin\theta_M + \cos\theta_H\cos\theta_M) + \frac{1}{2}(N_z M^2)\cos^2\theta_M$$
$$+ K_1\sin^2\theta_M + K_2\sin^4\theta_M$$

The in-plane static field orientation as Chappert assumed where $\theta'_H = \frac{\pi}{2}$ makes the orientation of the magnetization vector in equilibrium follow suit

$$\theta'_{M,0} = \frac{\pi}{2}$$

This leads to the following equilibrium position.

$$HM\sin\left(\phi'_H - \phi'_{M,0}\right) = (4\pi M^2 - 2K_1 - 4K_2)\sin\phi'_{M,0}\cos\phi'_{M,0}$$
$$+ 4K_2\sin^3\phi'_{M,0}\cos\phi'_{M,0}$$

Unfortunately there is no closed form solution for the orientation of the magnetization from this expression. In order to solve for the magnetization we must impose a specific static field orientation. In experimental setups with arbitrary static fields the out-of-plane angle is set experimentally or used as a fitting parameter.[48]

Without a perturbing rf field this result leads to a precession cone half angle given by the following expression:

$$\sin^2 \alpha = \frac{4\pi M^2 - 2K_1}{4K_2}$$

where α is associated with the azimuth angle $\phi'_{M,0}$ by the relation,

$$\alpha = \frac{\pi}{2} - \phi'_{M,0}$$

As mentioned before the general result for the SB method in spherical coordinates is as follows

$$\left(\frac{\omega_0}{\gamma}\right)^2 = \frac{1}{M^2 \sin^2 \theta_{M,0}} \left[F_{\phi_M \phi_M} F_{\theta_M \theta_M} - F^2_{\phi_M \theta_M}\right]$$

Using the SB method, Chappert discovered that the resonant frequency was related to the geometric average of two virtual fields in the following way, similar to the theoretical groundwork by Cronemeyer et. al.[47], in the following way,

$$\left(\frac{\omega_0}{\gamma}\right)^2 = H_1 H_2$$

where

$$H_1 = \left[H \cos\left(\phi'_H - \phi'_{M,0}\right) - \left(4\pi M - \frac{2K_1}{M} - \frac{4K_2}{M}\right) \sin^2 \phi'_{M,0} - \frac{4K_2}{M} \sin^4 \phi'_{M,0}\right]$$

$$H_2 = \left[H \cos\left(\phi'_H - \phi'_{M,0}\right) + \left(4\pi M - \frac{2K_1}{M} - \frac{4K_2}{M}\right) \cos\left(2\phi'_{M,0}\right) + \frac{4K_2}{M}\left(3 \sin^2 \phi'_{M,0} \cos^2 \phi'_{M,0} - \sin^4 \phi'_{M,0}\right)\right]$$

This process can be seen in detail in the Appendix C

Schulz and Baberschke used the same general form of the geometric average of two virtual fields based in Vonsovskii coordinates.[18]

Appendix D shows that Chappert's results match Okada's results which are expressed in Vonsovskii coordinates.[48] It is reassuring to see agreement with results between two researchers using different coordinate systems.

Okada, who worked in SI units, used the following expressions for the free energy, the resonant frequency, and the virtual magnetic fields, which all agree under the same conditions imposed by Chappert.

$$F = -HM\cos(\theta_H - \theta_M) + \frac{M^2}{2\mu_0}\cos^2\theta_M + K_1\sin^2\theta_M + K_2\sin^4\theta_M$$

$$F = -HM\cos(\theta_H - \theta_M) - \left(-\frac{M^2}{2\mu_0} + K_1 + 2K_2\right)\cos^2\theta_M + K_2\cos^4\theta_M + K_1 + K_2$$

$$f = \frac{g\mu_0\mu}{2\pi\hbar}\sqrt{H_1H_2}$$

$$\omega = \gamma\mu_0\sqrt{H_1H_2}$$

$$H_1 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} + \frac{2K_1}{M} + \frac{4K_2}{M}\right)\cos^2\theta_M - \frac{4K_2}{M}\cos^4\theta_M$$

$$H_2 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} + \frac{2K_1}{M} + \frac{4K_2}{M}\right)\cos 2\theta_M - \frac{2K_2}{M}(\cos 2\theta_M + \cos 4\theta_M)$$

2.5 BASELGIA'S DERIVATION AND CORRECTION

Since the first article by Kittel in 1947[33], methods of finding the FMR frequency have gradually increased in complexity by different authors weakening the restrictions for their validity. In 1988, Baselgia et. al.[49] discovered a mathematical subtlety in the earlier SB method. They uncovered that the distinct symmetry properties of the different terms in the free energy F are not visible in the expression of the SB method for finding the resonance frequency. They found that the discrepancy is due to the mixing of the Zeeman and other terms in the SB result for the resonance frequency. Baselgia et. al. show that using the SB result with a cubically symmetric system, a rotation of the effective field one way produces a different resonant frequency than another way. This result is unphysical since the resonant frequency should be invariant under rotations by $\frac{\pi}{2}$in a cubically symmetric system.

As mentioned previously the general result for the SB method in spherical coordinates is as follows

$$\left(\frac{\omega_0}{\gamma}\right)^2 = \frac{1}{M^2 \sin^2 \theta_{M,0}}\left[F_{\phi_M \phi_M} F_{\theta_M \theta_M} - F^2_{\phi_M \theta_M}\right]$$

In order to ensure rotational invariance of the resonant frequency, Baselgia et. al. modified the SB result by adding terms dependent on the first derivatives of the free energy. These terms take into account the effect of nonequilibrium dynamics which reflect the fact that the magnetization orientation is not exactly the same as the effective field orientation.

$$\left(\frac{\omega}{\gamma}\right)^2 = \frac{1}{M^2}\left[F_{\theta_M \theta_M}\left(\frac{F_{\phi_M \phi_M}}{\sin^2 \theta_M} + \frac{\cos \theta_M}{\sin \theta_M} F_{\theta_M}\right) - \left(\frac{F_{\theta_M \phi_M}}{\sin \theta_M} + \frac{\cos \theta_M}{\sin \theta_M}\frac{F_{\phi_M}}{\sin \theta_M}\right)^2\right]$$

In the definition of the free energy, Baselgia et. al. include uniaxial anisotropy energy terms taken out to sixth order with respect to the directional cosines, α. However, Baselgia et. al. absorbed the second order uniaxial anisotropy terms into the shape anisotropy term, which both depend on α^2 creating an effective shape anisotropy tensor $\overleftrightarrow{N}^{eff}$.

$$\begin{aligned} F = -\vec{H}\cdot\vec{M} + \frac{1}{2}\left(N_x^{eff} M_x^2 + N_y^{eff} M_y^2 + N_z^{eff} M_z^2\right) + K_{(1)}^{cub}\left(m_x^2 m_y^2 + m_x^2 m_z^2 + m_y^2 m_z^2\right) \\ + K_{(2)}^{cub}\, m_x^2 m_y^2 m_z^2 \end{aligned}$$

where $m_i = \frac{M_i}{M}$, and as mentioned second order terms $\frac{1}{2} K_i m_i^2$ are merged into N_i^{eff}.

We arrive at the conditions for equilibrium by equating the first derivatives of the free energy with zero.

$$\begin{aligned}
0 = & -H_x \cos\theta_M \cos\phi_M - H_y \cos\theta_M \sin\phi_M + H_z \sin\theta_M \\
& + \left(MN_x^{eff} \cos^2\phi_M + MN_y^{eff} \sin^2\phi_M - MN_z^{eff}\right) \cos\theta_M \sin\theta_M \\
& + \frac{2K_{(1)}^{cub}}{M} [2 \cos\theta_M \sin^3\theta_M \cos^2\phi_M \sin^2\phi_M \\
& + \cos\theta_M \sin\theta_M (\cos^2\theta_M - \sin^2\theta_M)] \\
& + \frac{2K_{(2)}^{cub}}{M} \cos\theta_M \sin^3\theta_M \cos^2\phi_M \sin^2\phi_M (2\cos^2\theta_M - \sin^2\theta_M)
\end{aligned}$$

$$\begin{aligned}
0 = & \; H_x \sin\phi_M - H_y \cos\phi_M + \left(MN_y^{eff} - MN_x^{eff}\right) \sin\theta_M \cos\phi_M \sin\phi_M \\
& + \frac{2K_{(1)}^{cub}}{M} \sin^3\theta_M \cos\phi_M \sin\phi_M (\cos^2\phi_M - \sin^2\phi_M) \\
& + \frac{2K_{(2)}^{cub}}{M} \cos^2\theta_M \sin^3\theta_M \cos\phi_M \sin\phi_M (\cos^2\phi_M - \sin^2\phi_M)
\end{aligned}$$

Unfortunately, similar to Chappert's derivation, it is not possible to evaluate the equilibrium angles of θ_M and ϕ_M from the equilibrium equations for an arbitrary direction of the applied field $\vec{H}$ in closed form. Therefore approximation techniques need be used such as those developed by He et. al.[50] Once the equilibrium angles $\theta_{M,0}$ and $\phi_{M,0}$ have been evaluated the equation for the resonance frequency can be expressed.

$$
\begin{aligned}
\left(\frac{\omega}{\gamma}\right)^2 = \Bigg(& H_x \sin\theta_M \cos\phi_M + H_y \sin\theta_M \sin\phi_M + H_z \cos\theta_M \\
& + \left(MN_x^{eff} \cos^2\phi_M + MN_y^{eff} \sin^2\phi_M - MN_z^{eff}\right)(\cos^2\theta_M - \sin^2\theta_M) \\
& + \frac{2K_{(1)}^{cub}}{M}[\cos^4\theta_M + \sin^4\theta_M (\cos^4\phi_M + \sin^4\phi_M) \\
& - 3\cos^2\theta_M \sin^2\theta_M (1 + \cos^4\phi_M + \sin^4\phi_M)] \\
& + \frac{2K_{(2)}^{cub}}{M}[\sin^2\theta_M \cos^2\phi_M \sin^2\phi_M (6\cos^4\theta_M + \sin^4\theta_M \\
& - 11\cos^2\theta_M \sin^2\theta_M)]\Bigg) \\
& \times \Bigg(H_x \sin\theta_M \cos\phi_M + H_y \sin\theta_M \sin\phi_M + H_z \cos\theta_M \\
& + MN_x^{eff}(\sin^2\phi_M - \sin^2\theta_M \cos^2\phi_M) \\
& + MN_y^{eff}(\cos^2\phi_M - \sin^2\theta_M \sin^2\phi_M) - MN_z^{eff}\cos^2\theta_M \\
& + \frac{2K_{(1)}^{cub}}{M}[\cos^4\theta_M + \sin^4\theta_M (\cos^4\phi_M + \sin^4\phi_M) \\
& - 6\sin^2\theta_M \cos^2\phi_M \sin^2\phi_M] \\
& + \frac{2K_{(2)}^{cub}}{M}\{\cos^2\theta_M \sin^2\theta_M [\cos^4\phi_M + \sin^4\phi_M \\
& - (4 + 3\sin^2\theta_M)\cos^2\phi_M \sin^2\phi_M]\}\Bigg) \\
& - \Bigg(\left(-MN_x^{eff} + MN_y^{eff}\right)\cos\theta_M \cos\phi_M \sin\phi_M \\
& + 3\frac{2K_{(1)}^{cub}}{M}[\cos\theta_M \sin^2\theta_M \cos\phi_M \sin\phi_M (\cos^2\phi_M - \sin^2\phi_M)] \\
& + \frac{2K_{(2)}^{cub}}{M}[\cos\theta_M \sin^2\theta_M (3\cos^2\theta_M \\
& - 2\sin^2\theta_M)\cos\phi_M \sin\phi_M (\cos^2\phi_M - \sin^2\phi_M)]\Bigg)^2
\end{aligned}
$$

This seemingly daunting equation possesses the benefit of superb generality. Not only are we able to predict FMR behavior at any orientation, the anisotropy factors are expanded out to sixth order. For our geometry and to fourth order in β this equation simplifies to the following:

$$\omega = \gamma\mu_0\sqrt{H_1 H_2}$$

$$H_1 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} - \frac{K_1}{M} - \frac{4K_2}{M}\right)\cos^2\theta_M - \frac{2K_2}{M}$$

$$H_2 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} - \frac{2K_1}{M}\right)\cos 2\theta_M + \frac{2K_2}{M}\cos 4\theta_M$$

Consequently, there is a discrepancy between the simplified Baselgia result and the Okada result:

$$H_1 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} + \frac{2K_1}{M} + \frac{4K_2}{M}\right)\cos^2\theta_M - \frac{4K_2}{M}\cos^4\theta_M$$

$$H_2 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} + \frac{2K_1}{M} + \frac{4K_2}{M}\right)\cos 2\theta_M - \frac{2K_2}{M}(\cos 2\theta_M + \cos 4\theta_M)$$

It is reassuring that the form of the equations are similar, however it is suspicious that the effect of voltage anisotropy is opposite with respect to each case. This discrepancy could be due to the symmetry correcting terms Baselgia added. This process is outlined in appendix E. We will adhere to the Okada result however due to the group's recent development of voltage anisotropy effects.

Additionally, Okada et. al. defined effective field contributions due to the effects of first and second order anisotropy terms in order to better grasp the effect compared with applied and demagnetizing fields as follows:

$$H_{K_1}^{\text{eff}} = -\frac{M}{\mu_0} + \frac{2K_1}{M} + \frac{4K_2}{M}$$

$$H_{K_2} = \frac{4K_2}{M}$$

2.6 SPIN TRANSFER TORQUE (STT) DRIVEN FMR

It has been established that a spin polarized electrical current can influence the magnetic state of a ferromagnetic material through spin-transfer torque.[5] FMR can be excited using this phenomenon. As in standard FMR, in an applied magnetic field, the magnetization of the sample precesses. As microwaves are delivered to the sample at the precession frequency resonance occurs. In the case of STT driven FMR, the microwaves are delivered in the form of an ac current. The effect of the spin polarized current acts to magnify the resonance condition increasing the precession amplitude. The process works in the following way, as potential is applied to the MTJ the current flowing through the device becomes spin polarized[51–53] thereby realigning the magnetization of the free layer. At a phase π later the spin polarized ac current forces the magnetization in another direction. This perturbation of the magnetization also has the effect of varying the resistance of the MTJ due to the misalignment of the magnetizations. However, the voltage rectification signal associated with this change in resistance includes overwhelming noise. Therefore, it is common practice to use a lock-in amplifier to increase the signal to noise ratio substantially.

Chapter 3: Experimental Setup

3.1 MTJs

Basic Operation

As mentioned before, a MTJ consists of two thin layers of magnetic material separated by a very thin layer of insulating material. The magnetization of one layer is free to be easily influenced by an applied magnetic field while the other is pinned by short range effects. The insulating material acts as an energy barrier to charge carriers that is tunable during fabrication by controlling the thickness. This energy barrier requires that the charge carriers tunnel through the insulating layer in order for current to flow. The resistance of the device depends on the relative alignment of the magnetizations of the free and pinned layers, which can be influenced by an applied magnetic field, spin polarized current, etc. If the magnetizations are parallel (antiparallel) the probability of electrons tunneling through the insulating layer is higher (lower) due to the higher (lower) density of states at the subsequent layer.

Fabrication

The MTJs studied in this work were produced at the University of Minnesota and have sizes on the order of 100 nm. They were grown using conventional magnetron sputtering and patterned using E beam lithography into ellipses with aspect ratios of about 2.5.The active layers in the device have the following structure and thicknesses: $Co_{60}Fe_{20}B_{20}$ (2.0 nm)//MgO (0.8 nm)//$Co_{40}Fe_{40}B_{20}$ (2.4 nm)/Ru (0.85 nm)/$Co_{70}Fe_{30}$ (2.5 nm)/PtMn(15 nm) surrounded by capping layers. The first CoFeB layer is the free layer; MgO composes the tunneling barrier; and the second CoFeB layer is the pinned layer which is an element of the synthetic antiferromagnet consisting of the CoFeB/Ru/CoFe layers.[54–56]

3.2 ELECTRONIC INSTRUMENT SETUP

The experimental setup consists of a MTJ in a 0.7 T variable magnitude applied magnetic field and various source and measurement instruments. A current source provides dc bias across the MTJ, of which 700 μA was not exceeded in any of the measurements discussed. Amplitude modulated microwaves with a 20% modulation depth are supplied to the sample using a rf generator. Frequency and power range of the microwaves used in this work are 4 to 12 GHz and 8–10 dBm, respectively. A bias tee combines the ac and dc current to the device. The combination of a preamplifier, a lock-in amplifier tuned to the rf amplitude modulation reference frequency, and a nanovoltmeter detects the rectification voltage across the MTJ.

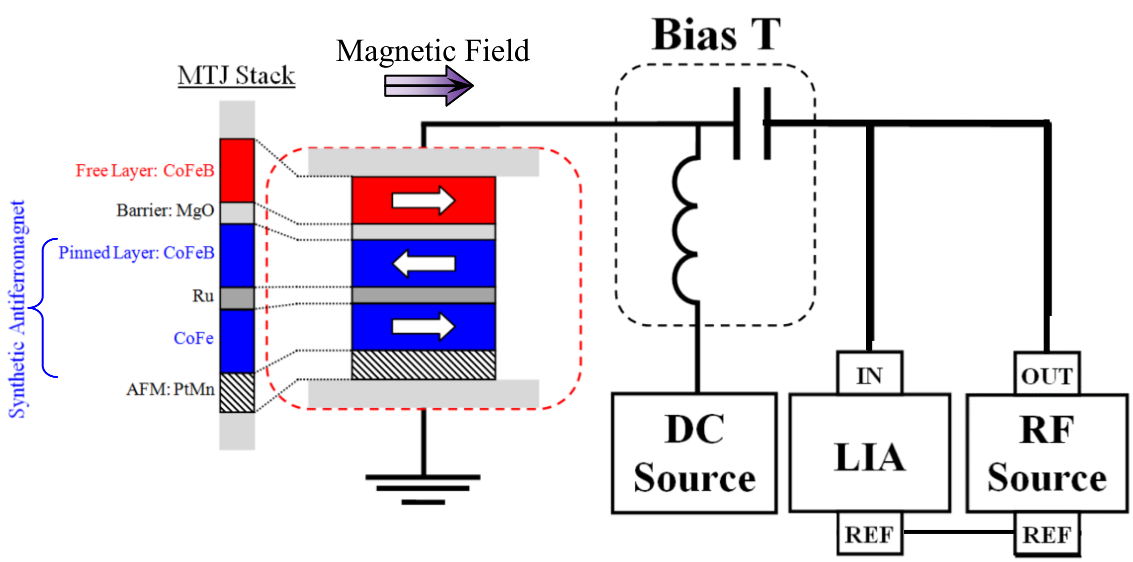

Figure 7: Experimental setup consisting of MTJ sample and measurement instruments including dual-phase lock-in amplifier (LIA).

3.3 COMPARISON OF MODULATION TECHNIQUES FOR FMR DETECTION

Heterodyne Detection of FMR

Conventional methods of detecting FMR consist of applying microwaves inside a cavity to excite resonance and a bolometer to detect absorbed radiation. In this work we use microwave currents at rf frequencies to excite resonance using spin injection and a heterodyne detection method requiring some form of modulation.[57] The heterodyne detection method works in the following way: a reference modulation signal is supplied from the modulation source directly to a phase-sensitive detector (PSD), in this case, a lock-in amplifier. An instantaneous voltage develops between the sample and ground due to the varying resistance while at constant current. The varying resistance results from the deviation of the magnetization orientation from its equilibrium position due to the applied rf current. The voltage signal and the reference signal are then processed by the lock-in amplifier, which multiplies the signals point by point using a digital mixer which separates the result into a dc signal and an ac signal at a frequency twice the modulation frequency. This process is modeled using the following equations.[58]

Where $V_{MTJ}\sin(\omega_{MTJ}t + \theta_{MTJ})$ represents the voltage signal from the MTJ sample and $V_{ref}\sin(\omega_{ref}t + \theta_{ref})$ represents the reference signal produced by the rf generator.

$$V_{LIA} = V_{MTJ}V_{ref}\sin(\omega_{MTJ}t + \theta_{MTJ})\sin(\omega_{ref}t + \theta_{ref})$$

$$V_{LIA} = \frac{1}{2}V_{MTJ}V_{ref}\cos([\omega_{MTJ} - \omega_{ref}]t + \theta_{MTJ} - \theta_{ref}) - \frac{1}{2}V_{MTJ}V_{ref}\cos([\omega_{MTJ} + \omega_{ref}]t + \theta_{MTJ} + \theta_{ref})$$

If $\omega_{MTJ} = \omega_{ref}$. This step is imposed using a phase-locked-loop. Which ensures both a fixed frequency ω and reference phase shift θ_{ref}.

$$V_{LIA} = \frac{1}{2} V_{MTJ} V_{ref} \cos(\theta_{MTJ} - \theta_{ref}) -$$
$$\frac{1}{2} V_{MTJ} V_{ref} \cos(2\omega t + \theta_{MTJ} + \theta_{ref})$$

A low pass filter is applied suppressing the signal at 2ω, leaving the dc signal designated as the rectification voltage.

$$V_{LIA} = \frac{1}{2} V_{MTJ} V_{ref} \cos(\theta_{MTJ} - \theta_{ref})$$

The result is a phase dependent dc signal proportional to the signal amplitude. To this point we have described a single-phase lock-in amplifier. In the absence of a second channel the reference phase must be shifted to match the signal phase in order to maximize the rectification signal. A dual-phase lock-in amplifier possesses two channels which execute these steps in parallel; however the second channel introduces a phase shift of 90°. In this way, a dual-phase lock-in amplifier has an X and Y channel which correspond to the in-phase and quadrature components of the signal, respectively.

$$X = V_{sig} \cos(\theta)\, Y = V_{sig} \sin(\theta)$$

Where θ is defined as the phase difference $\theta_{MTJ} - \theta_{ref}$.

This permits the maximization of the rectification signal at any phase, which is critical as the phase often changes during a measurement.

Two methods which can be used to supply the modulation required for heterodyne detection of FMR are amplitude modulation of the supplied microwaves (microwave AM) and modulation of the magnitude of the applied magnetic field (B-field modulation). It is important to note that every measurement in this work excluding this section was observed using the microwave AM technique.

The B-field modulation technique for investigating FMR has the benefit of detecting resonances while not varying power levels supplied to the sample, which is a complication associated with the microwave AM detection technique. As the amplitude of the microwaves change the power supplied to the sample changes as well. This introduces significant noise in any bolometric measurements preventing corroboration with heterodyne measurements.

Experimental Apparatus Modification Enabling B-field Modulation

The apparatus modification necessary to execute the B-field modulation technique called for the construction of a set of smaller magnetic field coils housed between the larger static applied field coils. The small coils were required to conservatively produce a 10 Gauss modulated field in the region of the sample. Incidentally, our design was capable of 113 Gauss B-field modulation at maximum ampacity rating.

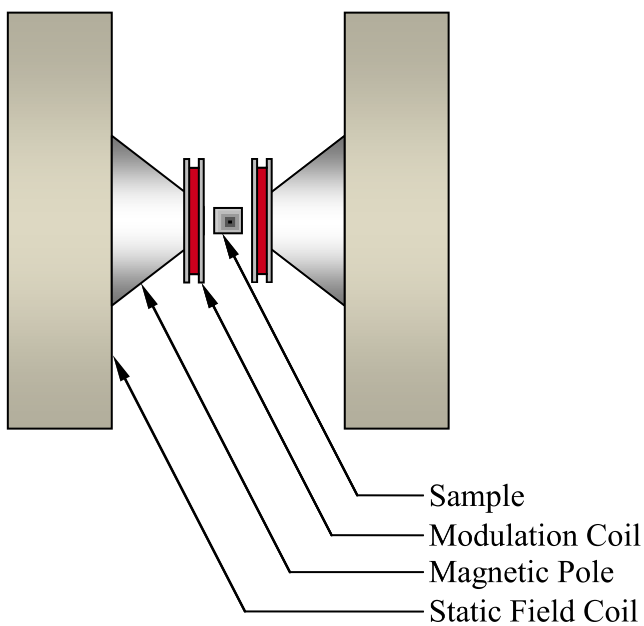

Figure 8: Experimental apparatus modification consisted of adding modulation coils near the sample region between the static field coils.

Frequency Dependent FMR Measurement Comparison

In an effort to understand the benefits and drawbacks of the B-field modulation method we endeavored to compare it with our established microwave AM detection method. We show the results below of the frequency dependent FMR measurements, which were duplicated as closely as possible between detection techniques.

In order to establish a benchmark FMR measurement upon which to compare the two techniques, we initiated a measurement covering the experimental parameters of magnetic field and frequency while measuring the rectification voltage produced by FMR. The microwave AM parameters were 20% amplitude depth at a frequency of 570.5 Hz. The B-field was modulated with an amplitude of 38 Gauss at a frequency of 363.53 Hz. We arrived at this frequency by conducting a signal to noise ratio investigation covering 120 Hz to 1500 Hz. This range was chosen due to the likely significant noise produced by power generation at low frequencies and the cutoff frequency of the bias tee on the DC side of the experimental setup. Wanting to evade harmonics of commonly used frequencies, we found a local maximum of the signal to noise ratio at 363.53 Hz.

The measurement procedure is as follows: under a constant bias of 10 μA the applied in-plane magnetic field is swept from -110 mT to 110 mT then back; the frequency increases by 100 MHz from 4 GHz to 12 GHz at a constant power of 10 dBm; and the field is swept at each of the frequencies. Measurement resolution was 81 points in frequency and 323 in B-field.

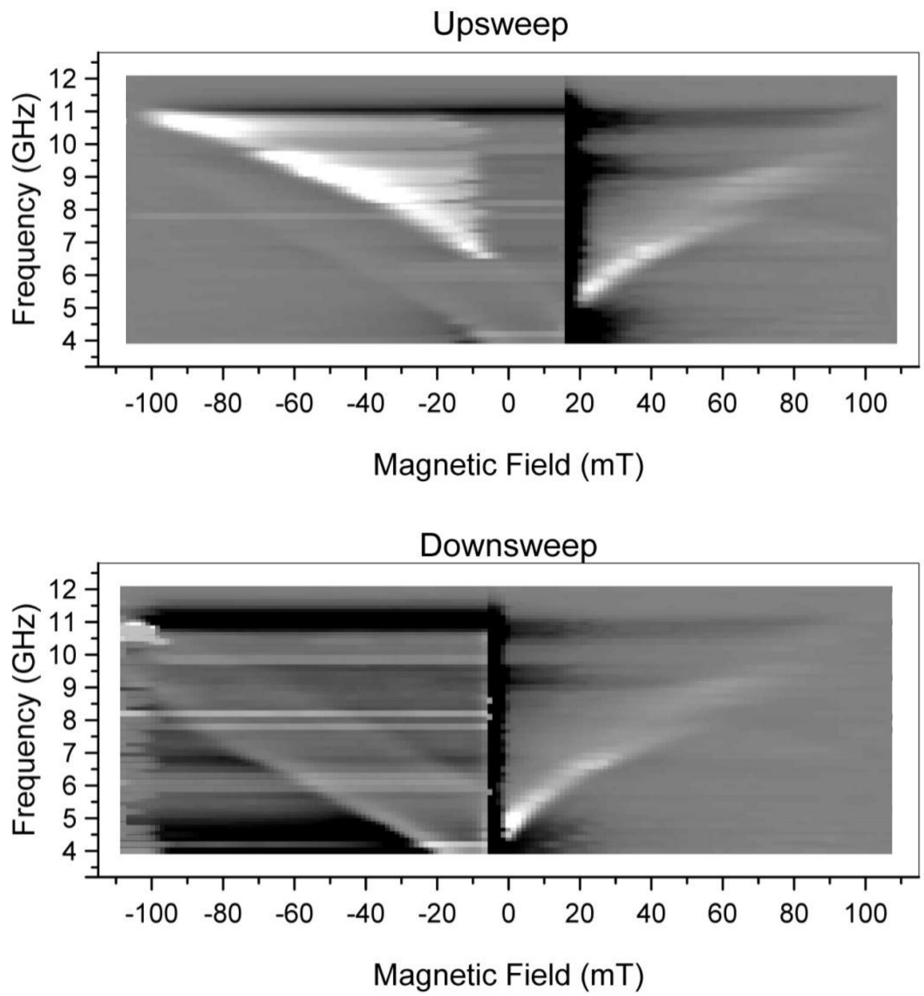

Figure 9: Rectification signal associated with the measurement using the microwave amplitude modulation detection method for the upsweep and downsweep of the magnetic field. White (Black) represents positive (Negative) voltage.

Due to the combination of the small resonance signal and the large jump in the rectification voltage at the point of switching, the zeroth order procedure for subtracting the background at negative fields fails to show detail at positive fields and vice versa. To solve this problem, background was subtracted from positive and negative fields separately then stitched together leaving the artifact of a stark line near zero field.

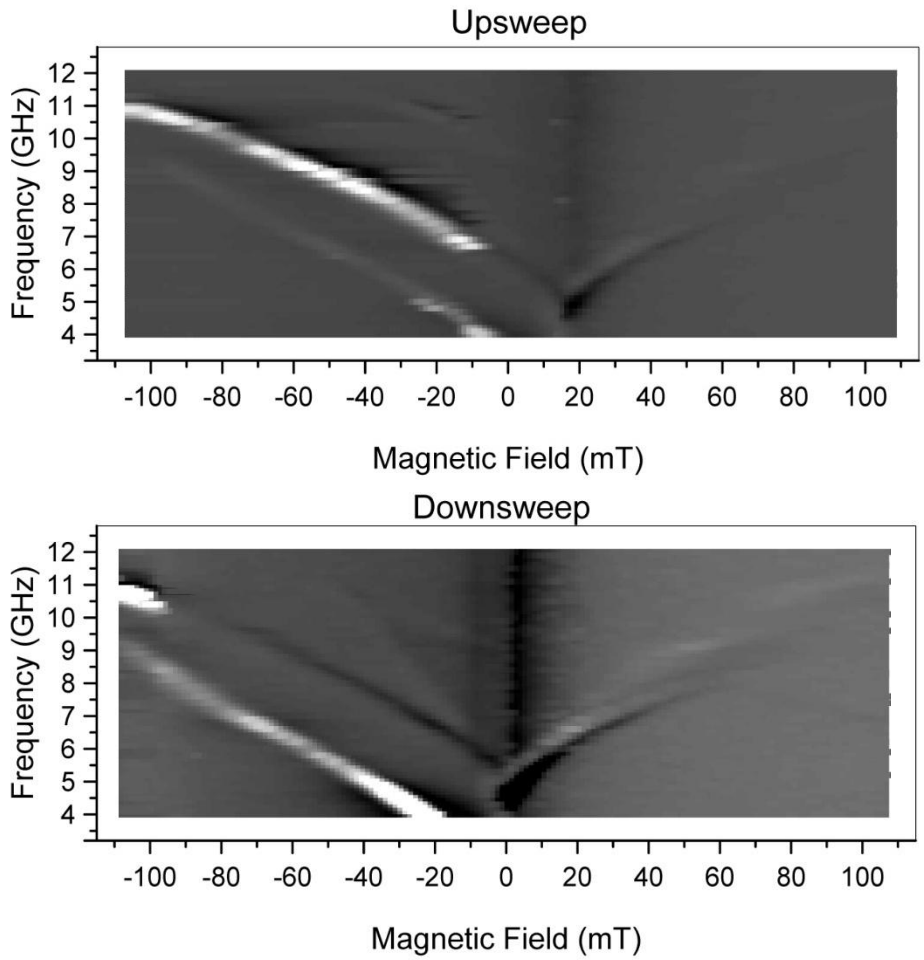

Figure 10: Rectification signal associated with the measurement using the B-field modulation detection method for the upsweep and downsweep of the magnetic field. White (Black) represents positive (Negative) voltage.

The measurement under B-field modulation produces a positive/negative voltage resonance signal that resembles the slope of the microwave AM signal. Incidentally, as we vary the B-field slightly we are measuring dV/dB, or the derivative of the microwave AM rectification voltage with respect to the magnetic field.

Single Sweep Comparison

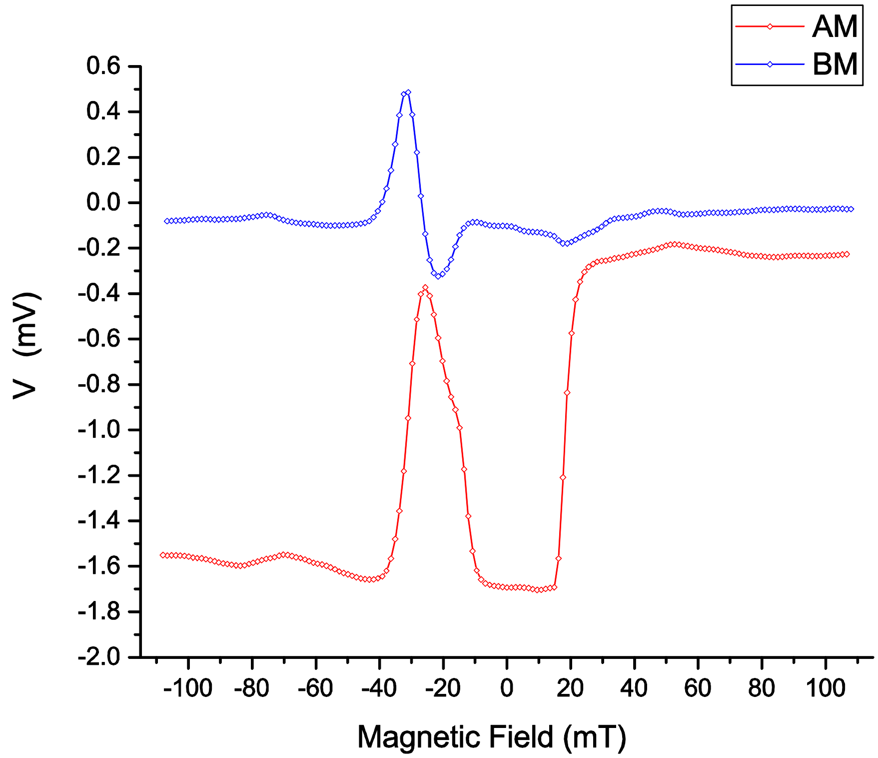

Figure 11: Lock-in amplifier detected rectification voltage: microwave amplitude modulation vs B-field modulation at 8GHz

Above we can see the individual sweeps at 8 GHz of the background subtracted rectification signal taken using the microwave AM and B-field modulated techniques. Notice the large step function like feature in the microwave AM signal which coincides with the free layer switching. This is the feature complicating the background subtraction process mentioned above. Note that the microwave AM signal appears to be twice as strong as the B-field modulated signal. Should any attempts be made to compensate the amplitude of the B-field modulated signal by increasing the ac B-field amplitude, one risks widening the FMR feature thereby lowering resolution.

In conclusion, the B-field modulated technique permits the measurement of the power reflected from the device with high accuracy and provides a clean background for analysis of the rectification signal. The microwave AM technique provides large signals, but unfortunately disrupts the accurate measurement of reflected power.

Chapter 4: Results

4.1 MR, FMR CURVES

Microwave amplitude modulated STT driven FMR produces specific spectra as shown below in figure 12 at the relatively small current bias of 10 μA. In this measurement, frequency is swept from 5 to 13.6 GHz with a magnetic field range from -270 mT to 270 mT.

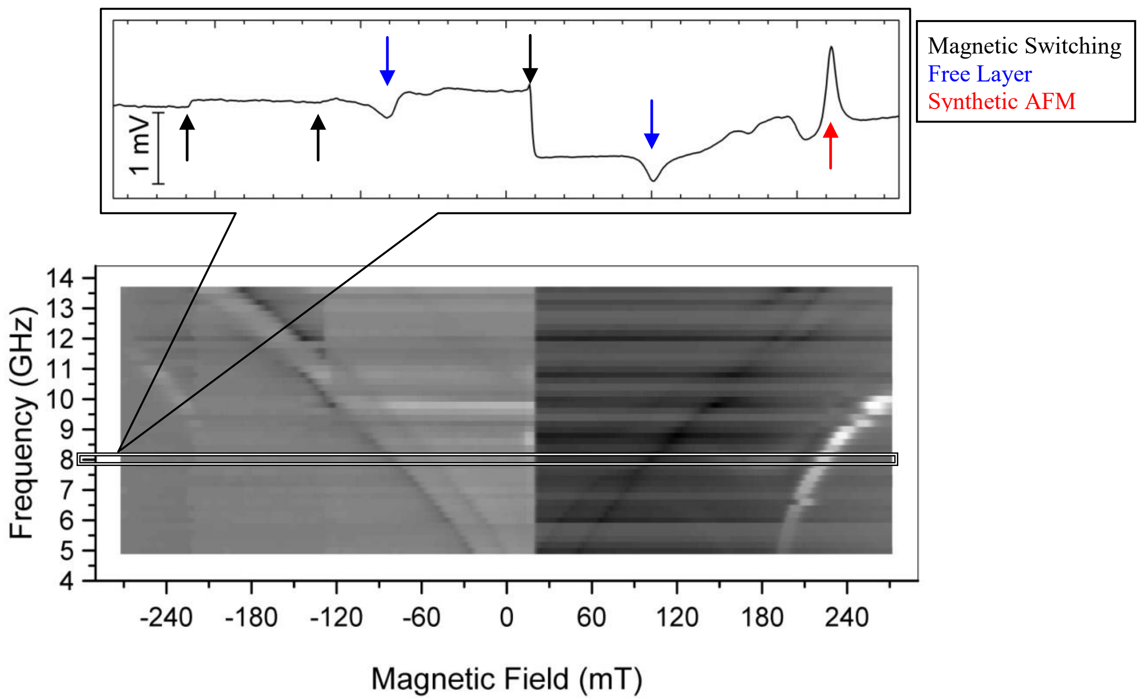

Figure 12: FMR spectra at 10 μA and 8 GHz for in-plane applied field orientation. Features of magnetic switching, free layer and synthetic AFM layer resonances are highlighted.

Using the frequency dependent data (figure 12) taken at an in-plane orientation (zero degrees) we are able to discriminate which features are the result of FMR excitation or switching events etc. Under these conditions, FMR is frequency dependent according to Kittel's equation, $\omega_0 = \gamma\sqrt{H(H+M)}$ (in SI and all units in Tesla). We are able to distinguish two separate resonance features at positive field, one with negative amplitude (black) and one with positive (white). At low frequencies, one originates near zero field and the other is offset by 190 mT. This offset is manifested by the exchange bias baked into the layers which compose the synthetic antiferromagnet; the associated feature signifies the resonance of the pinned layer. Therefore, it stands to reason that the feature centered near zero field is the free layer. The behavior of the resonance peak should theoretically curve at low frequencies then evolve towards a linear dependence as frequencies and fields rise in magnitude, which is displayed accordingly.

Using another device of the same construction, we investigate the resonance behavior and current bias dependence. Frequency dependent data give us the key to discern the resonance peaks residing in the single traces as shown below in figure 13. Magnetoresistance also has a role to play due to the correlation of the resonance signal and the resistance of the device. Closely inspecting the microwave AM versus B-field modulated measurements previously examined reveals an abrupt rise in resonance amplitude which coincides with the device switching to a higher resistance state. Incidentally, magnetoresistance, or rather the magnetoresistive ratio (MR) is the one of the most important metrics in the application of spintronic devices as it provides the contrast in bit discrimination. For the data currently under examination the magnetoresistance ratio (MR) is about 40%. In this particular device construction, we can interpret the orientation of the magnetizations of the respective free and pinned layers by closely examining the MR traces. At zero field (consumer and industrial applications' most likely default condition) the MTJ is in the low resistance state at around 425 Ω. At a low positive field (~20 mT) the MTJ switches to the high resistance state, around 600 Ω, and thus the two most important states for applications' sake are exhausted. Theoretically, this low positive field should be zero if we have a perfectly unhindered free layer. However there is hysteresis of the applied magnetic field on this order. Furthermore, there is a richer behavior outside this small range of field. At moderately high negative fields which coincide with the magnitude of the local bias exchange field we see a change to an intermediate resistance value. In the low (high) resistance state, the magnetization vectors of the free and pinned layers are parallel (antiparallel). In the intermediate resistance state the magnetization vectors are antiparallel, however the pinned layer and the nearest layer of the synthetic antiferromagnet are parallel. At high fields both positive and negative, all layers become saturated gradually, and therefore all are parallel.

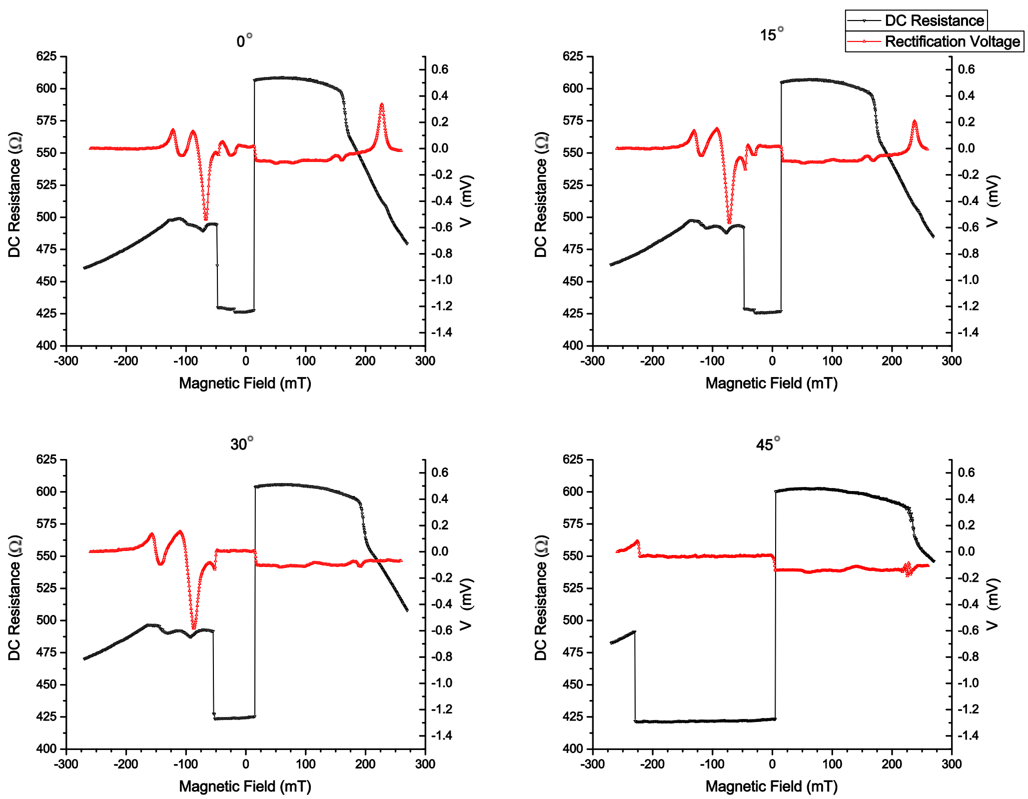

Figure 13: Magnetoresistance traces (left axis) and FMR spectra (right axis) at 20 μA and 8 GHz taken with an applied field orientation of 0°, 15°, 30°, and 45°.

4.2 BIAS AND ANGULAR DEPENDENCE

Current bias and angular dependent measurements were taken with currents from -700 μA to 700 μA, and at applied fields with a range from -270 mT to 270 mT. As shown in figure 15, DC current bias affects the behavior of the MTJ including shifting the resonance and switching fields, and augmenting the resonance amplitude. Pertaining to the main topic of this work, the resonance field is shifted symmetrically about zero bias in what seems to be a quadratic manner. We attribute this shift to most likely VCMA acting to decrease the resonance field magnitude. Near zero field the DC bias stimulates a field-like spin torque effect acting to shift the switching field linearly to higher positive (negative) fields at positive (negative) current biases. This measurement was taken with a resolution of 36 points of current and 800 points of magnetic field. At negative fields FMR is excited but is convolved with DC switching effects. The measurement frequency was optimized to focus on positive field FMR putting the resonance peaks on a relatively flat background. We can see that for the pinned layer at higher positive field, beyond 15° the resonance condition cannot be established due to the insufficient magnetic field range.

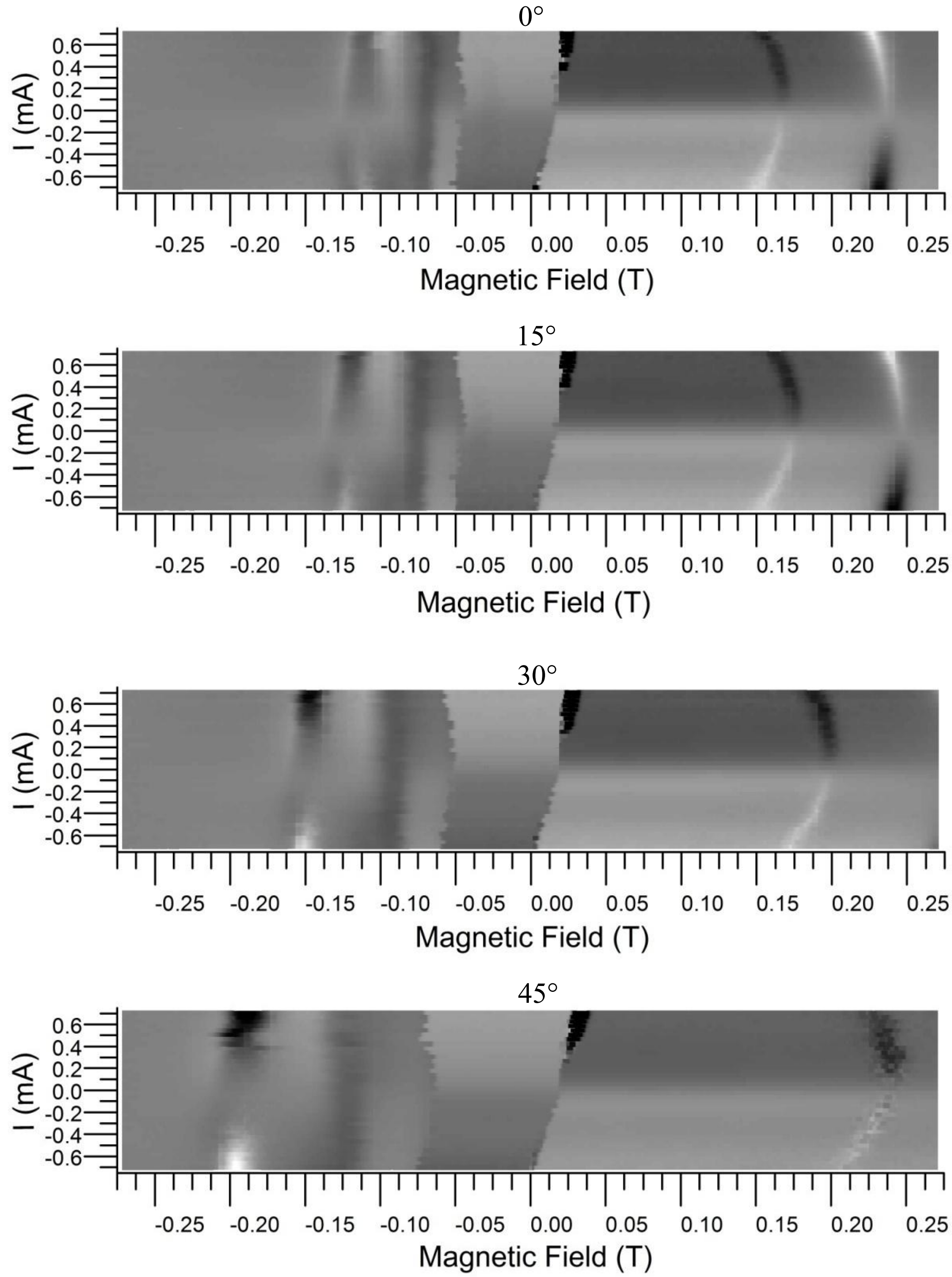

Figure 14: Rectification voltage signal depicting resonance spectra for out-of-plane angles of 0°, 15°, 30°, and 45°.

Chapter 5: Analysis

4.1 BIAS AND ANGULAR DEPENDENT FMR

Peaks/dips in the rectification voltage signal from the lock-in amplifier were extracted using a simple algorithm comparing the neighboring points to the data point in question. If the examined data point was higher (lower) than its neighbors it was labeled as a peak (dip). We did develop a statistically more accurate method (described below) of extracting the peak, width, and amplitude of the FMR features using a Lorentzian function as a fitting equation, however the method lacked the automation necessary for dealing with the required 144 separate nonlinear fits. The simple peak extraction method was completed at each angle and current bias, then false positives were removed culminating with figure 15 seen below. Since the pinned layer resonance evades capture at this field range, we focus on analyzing the free layer resonance at positive field.

As the out-of-plane angle increases the FMR features spread in field magnitude and width. One can see that the width of the resonance peak at 45° is much larger than the width at 0°. This effect is simply due to the applied field vector needing to compensate in magnitude for its out-of-plane orientation. The in-plane measurement is essentially projected onto an oblique plane oriented at the appropriate angle having inverse cosine dependence. It is reasonable to assume that if the symmetric shift in the FMR feature is due to an effective field supplied by VCMA, then adding an additional out-of-plane field by changing the angle of the applied field should perturb it in a most likely additive way.

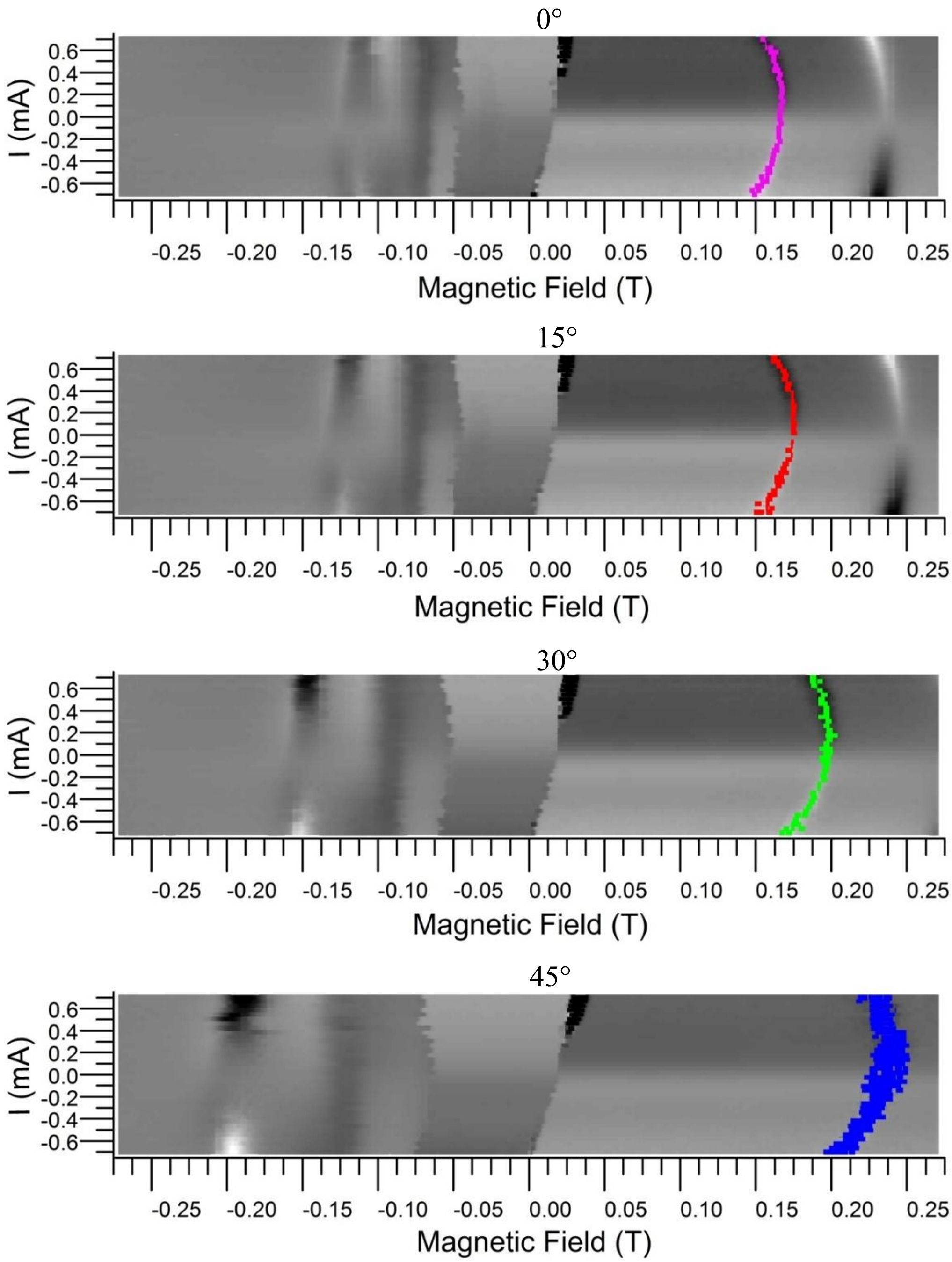

Figure 15: Bias dependence of rectification voltage for angles 0°,15°,30°,and 45°. FMR peaks/dips associated with the free layer excitation are extracted and highlighted.

4.2 FMR Feature Voltage Dependence

We plot the free layer FMR features as a function of voltage below in figure 16. The resonance field magnitude exhibits an inverse cosine behavior as a function of the out-of-plane angle. Also, the voltage dependent data seems to show a consistent quadratic behavior across all out-of-plane angles. A more subtle difference lies in what appears to be a shift in the center of the fit describing voltage dependent FMR peaks/dips from0.5 V to 0.1 V as the angle varies from 0º to 45º. It is not clear if this is an effect of adding an out-of-plane field in addition to the VCMA effective field, spin-transfer torque angular dependence, measurement variability, or current induced device modification.

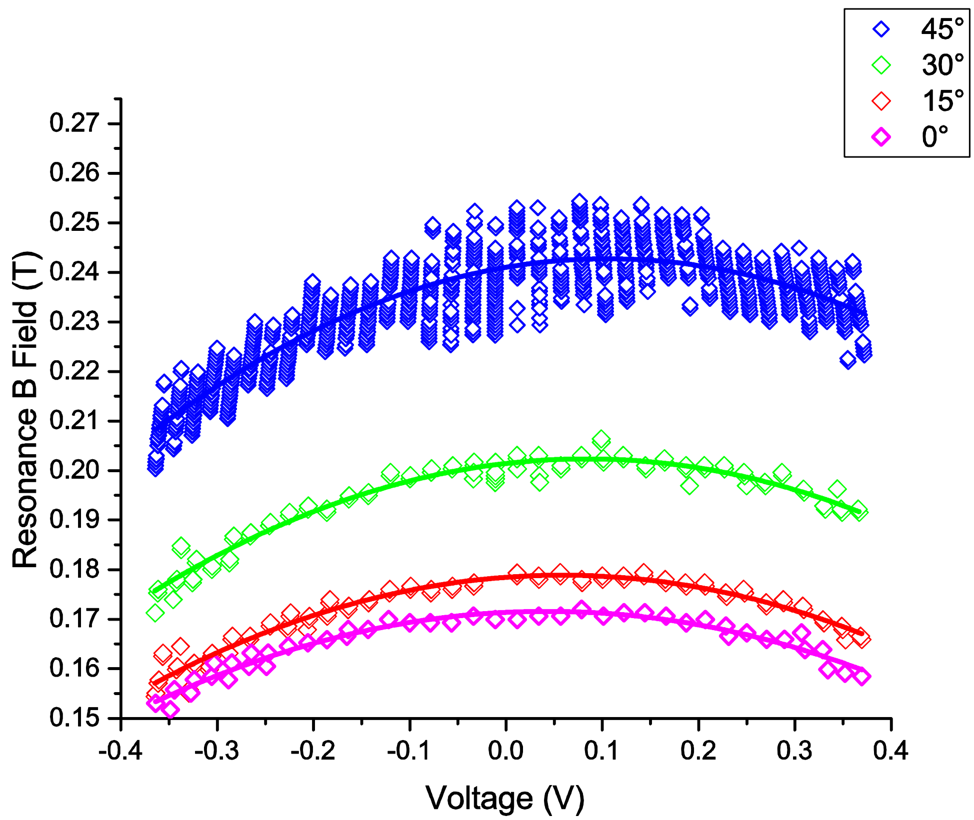

Figure 16: FMR peaks/dips and associated quadratic fits plotted as a function of voltage for out-of-plane angles 0°, 15°, 30°, and 45°.

One matter to be noted is the significant increase in width of the FMR feature at 45°. Although the real width of the feature did in fact increase, the width shown above may be exaggerated due to the peak finding procedure requiring a larger amount of neighbors to be effective. Regardless, one may notice firsthand the increase in variability of the rectification signal in the results section.

4.3 CALCULATED EFFECTIVE FIELD DUE TO VCMA

The electric potential dependence of the FMR features suggests the influence of voltage induced magnetic anisotropy which can be examined more closely using the FMR equations developed in the theory section. Assuming VCMA is causing the resonance field shift, we can use these equations to deduce the form of the magnetic anisotropy. First, we used the zero bias(assuming no VCMA effects), in-plane trace to solve for the saturation magnetization (304 mT) of the free layer in order to calibrate for the demagnetizing field. Then we applied the more general equations to the bias and angular dependent data. Every peak/dip extracted from the current bias dependent measurements was input as a data point in the equations:

$$\omega = \gamma\mu_0\sqrt{H_1 H_2}$$

$$H_1 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} + \frac{2K_1}{M}\right)\cos^2\theta_M$$

$$H_2 = H\cos(\theta_H - \theta_M) + \left(-\frac{M}{\mu_0} + \frac{2K_1}{M}\right)\cos 2\theta_M$$

Where the anisotropy factor K_1 was numerically solved for at each current bias and angle. We chose to use a modified version of Okada's equation with a simple first order dependence of VCMA to minimize complexity. When compared to the applied field, the magnitude of K_1 (being on the order of 10^5 when saturation magnetization is measured in Amperes per meter) does not elucidate or characterize the magnetic anisotropy induced by the applied bias voltage simply because of disparate units. Instead, we use an effective field combining demagnetizing field and VCMA as in Okada's work, albeit modified to include only first order anisotropy terms.

$$H_{K_1}^{eff} = -\frac{M}{\mu_0} + \frac{2K_1}{M}$$

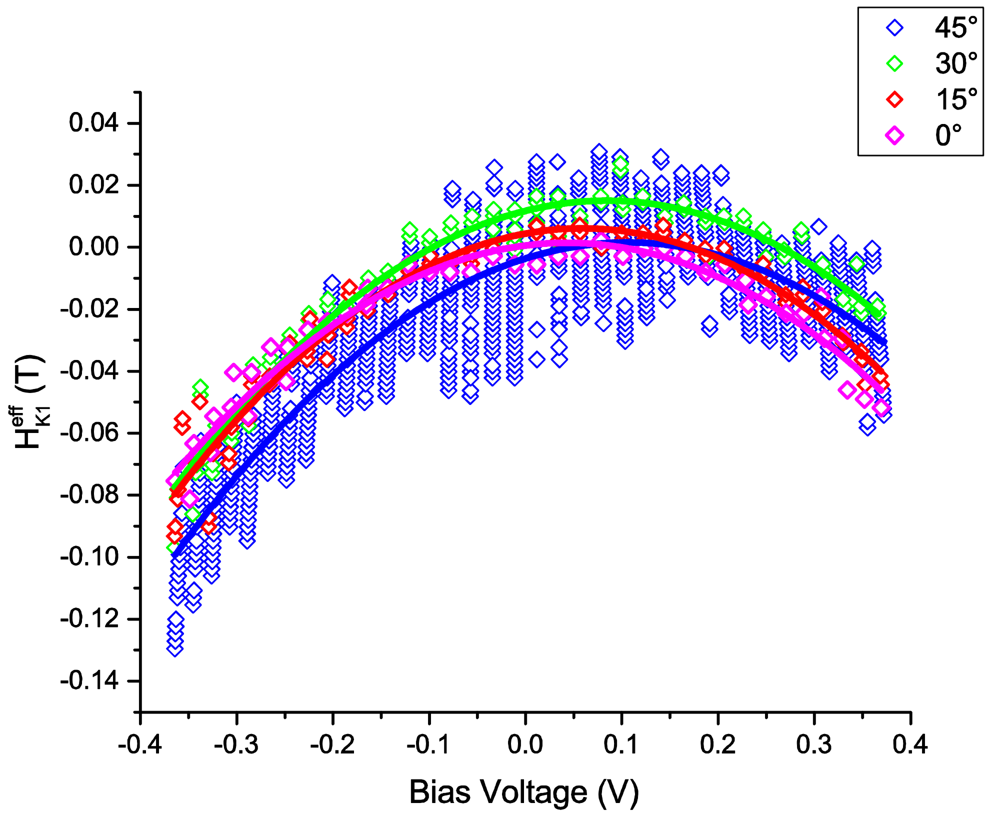

Figure 17: Effective field caused by voltage bias solved for using generalized angular dependent FMR equations with anisotropic corrective terms and fitted with a quadratic polynomial.

Voltage dependence of the magnetic anisotropy effective field shows clear quadratic dependence with a maximum magnitude of approximately 90 mT at a potential of -0.37 V. Furthermore, analysis has collapsed the data within measurement variation into the vicinity of a single curve. This signifies that apart from slightly shifting the center of the quadratic fit of the calculated effective field, the additive effect of an out-of-plane applied field at this magnitude does not significantly influence the VCMA effective field. This is interesting since the maximum out-of-plane applied fields for 0°,15°,30°, and 45° are 0 mT, 70 mT, 135 mT, and 191 mT, respectively. Clearly, out-of-plane fields for the higher angles are at least on par with the maximum anisotropic effective field of 90 mT.

4.4 LORENTZIAN FIT

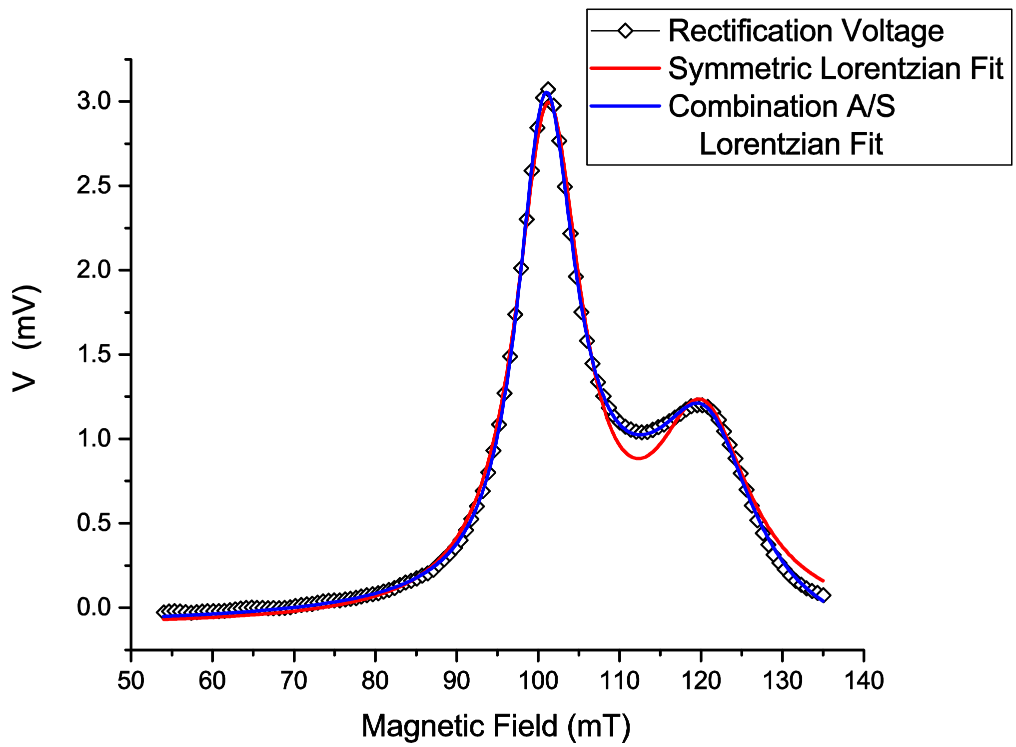

Figure 18: Microwave amplitude modulated rectification signal of FMR peaks of two layers with slightly different resonance B-fields. Red depicts the nonlinear fit using a single symmetric Lorentzian for each peak. Blue represents the nonlinear fit using an antisymmetric/symmetric Lorentzian pair for each peak.

As mentioned above, a peak finding technique using a Lorentzian function was developed that accurately deduces the position, amplitude and width of the

FMR feature. However, the technique was initially developed for a single tailored trace. And faced with the plethora of traces at various currents and angles, it would have been inefficient to compute the fits one by one in this manner. Attempts at automating the process taxed Mathematica to the point where solutions possessing reasonable computational time were beyond the skills of the author.

Nevertheless, figure 18 shows that although a simple symmetric Lorentzian fit can suitably estimate the peak position, amplitude, and width of the two FMR peaks, a combination antisymmetric/symmetric Lorentzian fit is superior.

Chapter 6: Summary

In this work we investigated the bias current dependence of FMR at 8 GHz using in-plane MTJs with cross sections on the order of 100 nm. We used angular dependent FMR equations that account for demagnetizing field and first order magnetic anisotropy to fit the bias dependent data. We found quadratic dependence of the proposed VCMA effective field with respect to voltage bias in the free layer. A maximum out-of-plane effective field of 90 mT was reached at a potential of 0.37 V. We also compared two measurement techniques of detecting FMR: amplitude modulation of the supplied microwaves and applied magnetic field modulation using smaller secondary coils between the static field coils. We found that the microwave amplitude modulation method provided a stronger rectification signal, while the B-field modulation method supplied a clean background. Lastly, a fitting procedure was developed to extract FMR feature resonance field, amplitude, and width, which revealed that a combination symmetric/antisymmetric Lorentzian structure provided a superior fit compared to a symmetric Lorentzian.

Appendix A

The conversion between Vonsovskii coordinates and Chappert coordinates was done by equating the appropriate components of the magnetization vector in their respective coordinates.

$$M_{x'} = M \sin\theta'_M \cos\phi'_M = M \sin\phi_M \sin\theta_M = M_y$$

$$M_{y'} = M \sin\theta'_M \sin\phi'_M = M \cos\theta_M = M_z$$

$$M_{z'} = M \cos\theta'_M = M \cos\phi_M \sin\theta_M = M_x$$

This leads to the basic relations

$$\sin\theta'_M \cos\phi'_M = \sin\phi_M \sin\theta_M$$

$$\sin\theta'_M \sin\phi'_M = \cos\theta_M$$

$$\cos\theta'_M = \cos\phi_M \sin\theta_M$$

Which we can manipulate to isolate each variable in terms of the other coordinate system using trigonometric relations. First, we start by isolating the Vonsovskii parameters in terms of Chappert parameters. Second, we do the converse.

$$\boxed{\cos\theta_M = \sin\theta'_M \sin\phi'_M}$$

$$\sin^2\theta_M = 1 - \cos^2\theta_M$$

$$\sin^2\theta_M = 1 - (\sin\theta'_M \sin\phi'_M)^2$$

$$\boxed{\sin\theta_M = \sqrt{1 - \sin^2\theta'_M \sin^2\phi'_M}}$$

$$\cos\phi_M = \frac{\cos\theta'_M}{\sin\theta_M}$$

$$\boxed{\cos\phi_M = \frac{\cos\theta'_M}{\sqrt{1 - \sin^2\theta'_M \sin^2\phi'_M}}}$$

$$\sin\phi_M = \frac{\sin\theta'_M \cos\phi'_M}{\sin\theta_M}$$

$$\boxed{\sin\phi_M = \frac{\sin\theta'_M \cos\phi'_M}{\sqrt{1 - \sin^2\theta'_M \sin^2\phi'_M}}}$$

$$\boxed{\cos\theta'_M = \cos\phi_M \sin\theta_M}$$

$$\sin^2\theta'_M = 1 - \cos^2\theta'_M$$

$$\sin^2\theta'_M = 1 - (\sin\theta'_M \sin\phi'_M)^2$$

$$\boxed{\sin\theta'_M = \sqrt{1 - \cos^2\phi_M \cos^2\theta_M}}$$

$$\cos\phi'_M = \frac{\sin\phi_M \sin\theta_M}{\sin\theta'_M}$$

$$\boxed{\cos\phi'_M = \frac{\sin\phi_M \sin\theta_M}{\sqrt{1 - \cos^2\phi_M \cos^2\theta_M}}}$$

$$\sin\phi'_M = \frac{\cos\theta_M}{\sin\theta'_M}$$

$$\boxed{\sin\phi'_M = \frac{\cos\theta_M}{\sqrt{1 - \cos^2\phi_M \cos^2\theta_M}}}$$

The conversion of the effective field vector is more limited due to the restriction Chappert imposed, $\theta'_H = \frac{\pi}{2}$.

$$\cos\theta_H = \sin\theta'_H \sin\phi'_H$$

$$\boxed{\cos\theta_H = \sin\phi'_H}$$

$$\sin^2\theta_H = 1 - \cos^2\theta_H$$

$$\sin^2\theta_H = 1 - (\sin\phi'_H)^2$$

$$\sin\theta_H = \sqrt{1 - \sin^2\phi'_H}$$

$$\boxed{\sin\theta_H = \cos\phi'_H}$$

$$\cos\phi_H = \frac{\cos\theta'_H}{\sin\theta_H}$$

$$\cos\phi_H = \frac{(0)}{\sin\theta_H}$$

$$\boxed{\cos\phi_H = 0}$$

$$\sin\phi_H = \frac{\cos\phi'_H}{\sin\theta_H}$$

$$\sin\phi_H = \frac{\cos\phi'_H}{\cos\phi'_H}$$

$$\boxed{\sin\phi_H = 1}$$

$$\cos\theta'_H = \cos\phi_H \sin\theta_H$$

$$\boxed{0 = (0)\sin\theta_H}$$

$$\sin^2\theta'_H = 1 - \cos^2\theta'_H$$

$$\sin^2\theta'_H = 1 - (0)^2$$

$$\boxed{1 = 1}$$

$$\cos\phi'_H = \frac{\sin\phi_M \sin\theta_H}{\sin\theta'_H}$$

$$\cos\phi'_H = \sin\phi_M \sin\theta_H$$

$$\boxed{\cos\phi'_H = \sin\theta_H}$$

$$\sin\phi'_H = \frac{\cos\theta_H}{\sin\theta'_H}$$

$$\boxed{\sin\phi'_H = \cos\theta_H}$$

Appendix B

Conversion of Free Energy F from Vonsovskii Coordinates to Chappert Coordinates

In order to show the equivalence of the free energy in both Chappert coordinates and Vonsovskii coordinates we start with the expression of the free energy in Vonsovskii coordinates which are the most general.

Notation :

Due to the way Mathematica works with subscripts the variables normally expressed with subscripts will simply be expressed by the following letter of the subscript. i.e.

$\theta_M = \theta M;\ \phi_M = \phi M;\ \theta_H = \theta H;\ \phi_H = \phi H;$

Also variables followed by a c denote that coordinate is in the Chappert coordinate system

```
ϕH =.; θH =.; ϕM =.; θM =.;

ϕHc =.; θHc =.; ϕMc =.; θMc =.;
```

Free Energy in Vonsovskii Coordinates

$$F = -M\,H\,(\mathrm{Cos}[\phi H - \phi M]\ \mathrm{Sin}[\theta H]\ \mathrm{Sin}[\theta M] + \mathrm{Cos}[\theta H]\ \mathrm{Cos}[\theta M]) + \frac{1}{2} M^2 \left(\mathrm{Nx}\ \mathrm{Cos}[\phi M]^2\ \mathrm{Sin}[\theta M]^2 + \mathrm{Ny}\ \mathrm{Sin}[\phi M]^2\ \mathrm{Sin}[\theta M]^2 + \mathrm{Nz}\ \mathrm{Cos}[\theta M]^2\right) + \mathrm{K1}\ \mathrm{Sin}[\theta M]^2 + \mathrm{K2}\ \mathrm{Sin}[\theta M]^4;$$

Applying appropriate shape anisotropy factors

$$\mathrm{Nx} = 0;\ \mathrm{Ny} = 0;$$

$$F = -M\,H\,(\mathrm{Cos}[\phi H - \phi M]\ \mathrm{Sin}[\theta H]\ \mathrm{Sin}[\theta M] + \mathrm{Cos}[\theta H]\ \mathrm{Cos}[\theta M]) + \frac{1}{2} M^2\ \mathrm{Nz}\ \mathrm{Cos}[\theta M]^2 + \mathrm{K1}\ \mathrm{Sin}[\theta M]^2 + \mathrm{K2}\ \mathrm{Sin}[\theta M]^4;$$

Conversion Equations

$$\mathrm{Cos}[\theta M] = \mathrm{Sin}[\theta Mc]\ \mathrm{Sin}[\phi Mc];$$

$$\mathrm{Sin}[\theta M] = \sqrt{1 - \mathrm{Sin}[\theta Mc]^2\ \mathrm{Sin}[\phi Mc]^2}\ ;$$

$$\mathrm{Cos}[\phi M] = \frac{\mathrm{Cos}[\theta Mc]}{\sqrt{1 - \mathrm{Sin}[\theta Mc]^2\ \mathrm{Sin}[\phi Mc]^2}};$$

$$\mathrm{Sin}[\phi M] = \frac{\mathrm{Sin}[\theta Mc]\ \mathrm{Cos}[\phi Mc]}{\sqrt{1 - \mathrm{Sin}[\theta Mc]^2\ \mathrm{Sin}[\phi Mc]^2}};$$

$$\text{Cos}[\theta H] = \text{Sin}[\theta Hc]\,\text{Sin}[\phi Hc];$$

$$\text{Sin}[\theta H] = \sqrt{1 - \text{Sin}[\theta Hc]^2\,\text{Sin}[\phi Hc]^2};$$

$$\text{Cos}[\phi H] = \frac{\text{Cos}[\theta Hc]}{\sqrt{1 - \text{Sin}[\theta Hc]^2\,\text{Sin}[\phi Hc]^2}};$$

$$\text{Sin}[\phi H] = \frac{\text{Sin}[\theta Hc]\,\text{Cos}[\phi Hc]}{\sqrt{1 - \text{Sin}[\theta Hc]^2\,\text{Sin}[\phi Hc]^2}};$$

Trigonometric Identity

$$\text{Cos}[\phi H - \phi M] = \text{Cos}[\phi H]\,\text{Cos}[\phi M] + \text{Sin}[\phi H]\,\text{Sin}[\phi M];$$

Plugging in Trigonometric Identity

$$F = -M H\,((\text{Cos}[\phi H]\,\text{Cos}[\phi M] + \text{Sin}[\phi H]\,\text{Sin}[\phi M])\,\text{Sin}[\theta H]\,\text{Sin}[\theta M] + \text{Cos}[\theta H]\,\text{Cos}[\theta M]) + \frac{1}{2} M^2\,\text{Nz}\,\text{Cos}[\theta M]^2 + \text{K1}\,\text{Sin}[\theta M]^2 + \text{K2}\,\text{Sin}[\theta M]^4;$$

Plugging in conversion equations

$$F = -M H\left(\left(\frac{\text{Cos}[\theta Hc]}{\sqrt{1 - \text{Sin}[\theta Hc]^2\,\text{Sin}[\phi Hc]^2}}\,\frac{\text{Cos}[\theta Mc]}{\sqrt{1 - \text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2}} + \frac{\text{Sin}[\theta Hc]\,\text{Cos}[\phi Hc]}{\sqrt{1 - \text{Sin}[\theta Hc]^2\,\text{Sin}[\phi Hc]^2}}\,\frac{\text{Sin}[\theta Mc]\,\text{Cos}[\phi Mc]}{\sqrt{1 - \text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2}}\right)\sqrt{1 - \text{Sin}[\theta Hc]^2\,\text{Sin}[\phi Hc]^2}\,\sqrt{1 - \text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2} + \text{Sin}[\theta Hc]\,\text{Sin}[\phi Hc]\,\text{Sin}[\theta Mc]\,\text{Sin}[\phi Mc]\right) + \frac{1}{2} M^2\,\text{Nz}\,(\text{Sin}[\theta Mc]\,\text{Sin}[\phi Mc])^2 + \text{K1}\left(\sqrt{1 - \text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2}\right)^2 + \text{K2}\left(\sqrt{1 - \text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2}\right)^4;$$

Simplify[F]

$$\text{K1} + \text{K2} - H M\,\text{Cos}[\theta Hc]\,\text{Cos}[\theta Mc] - H M\,\text{Cos}[\phi Hc]\,\text{Cos}[\phi Mc]\,\text{Sin}[\theta Hc]\,\text{Sin}[\theta Mc] - H M\,\text{Sin}[\theta Hc]\,\text{Sin}[\theta Mc]\,\text{Sin}[\phi Hc]\,\text{Sin}[\phi Mc] - \text{K1}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 - 2\,\text{K2}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 + \frac{1}{2} M^2\,\text{Nz}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 + \text{K2}\,\text{Sin}[\theta Mc]^4\,\text{Sin}[\phi Mc]^4$$

$$\text{Collect}\Big[\text{K1} + \text{K2} - H M\,\text{Cos}[\theta Hc]\,\text{Cos}[\theta Mc] - H M\,\text{Cos}[\phi Hc]\,\text{Cos}[\phi Mc]\,\text{Sin}[\theta Hc]\,\text{Sin}[\theta Mc] - H M\,\text{Sin}[\theta Hc]\,\text{Sin}[\theta Mc]\,\text{Sin}[\phi Hc]\,\text{Sin}[\phi Mc] - \text{K1}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 - 2\,\text{K2}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 + \frac{1}{2} M^2\,\text{Nz}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 + \text{K2}\,\text{Sin}[\theta Mc]^4\,\text{Sin}[\phi Mc]^4,\ \text{K1}\Big];$$

$$\text{Collect}\Big[\text{K2} - H M\,\text{Cos}[\theta Hc]\,\text{Cos}[\theta Mc] - H M\,\text{Cos}[\phi Hc]\,\text{Cos}[\phi Mc]\,\text{Sin}[\theta Hc]\,\text{Sin}[\theta Mc] - H M\,\text{Sin}[\theta Hc]\,\text{Sin}[\theta Mc]\,\text{Sin}[\phi Hc]\,\text{Sin}[\phi Mc] - 2\,\text{K2}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 + \frac{1}{2} M^2\,\text{Nz}\,\text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2 + \text{K2}\,\text{Sin}[\theta Mc]^4\,\text{Sin}[\phi Mc]^4 + \text{K1}\left(1 - \text{Sin}[\theta Mc]^2\,\text{Sin}[\phi Mc]^2\right),\ \text{K2}\Big];$$

```
Collect[-H M Cos[θHc] Cos[θMc] - H M Cos[ϕHc] Cos[ϕMc] Sin[θHc] Sin[θMc] -
  H M Sin[θHc] Sin[θMc] Sin[ϕHc] Sin[ϕMc] + 1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 +
  K1 (1 - Sin[θMc]^2 Sin[ϕMc]^2) + K2 (1 - 2 Sin[θMc]^2 Sin[ϕMc]^2 + Sin[θMc]^4 Sin[ϕMc]^4), H]

1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 +
 H (-M Cos[θHc] Cos[θMc] - M Cos[ϕHc] Cos[ϕMc] Sin[θHc] Sin[θMc] - M Sin[θHc] Sin[θMc] Sin[ϕHc] Sin[ϕMc]) +
 K1 (1 - Sin[θMc]^2 Sin[ϕMc]^2) + K2 (1 - 2 Sin[θMc]^2 Sin[ϕMc]^2 + Sin[θMc]^4 Sin[ϕMc]^4)

F = -H M (Cos[θHc] Cos[θMc] + Cos[ϕHc] Cos[ϕMc] Sin[θHc] Sin[θMc] + Sin[θHc] Sin[θMc] Sin[ϕHc] Sin[ϕMc]) +
  1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 + K1 (1 - Sin[θMc]^2 Sin[ϕMc]^2) +
  K2 (1 - 2 Sin[θMc]^2 Sin[ϕMc]^2 + Sin[θMc]^4 Sin[ϕMc]^4)

F = -H M (Cos[θHc] Cos[θMc] + Cos[ϕHc] Cos[ϕMc] Sin[θHc] Sin[θMc] + Sin[θHc] Sin[θMc] Sin[ϕHc] Sin[ϕMc]) +
  1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 + -K1 Sin[θMc]^2 Sin[ϕMc]^2 -
  2 K2 Sin[θMc]^2 Sin[ϕMc]^2 + K2 Sin[θMc]^4 Sin[ϕMc]^4 + K1 + K2

F = -H M (Cos[θHc] Cos[θMc] + Cos[ϕHc] Cos[ϕMc] Sin[θHc] Sin[θMc] + Sin[θHc] Sin[θMc] Sin[ϕHc] Sin[ϕMc]) +
  1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 + - (K1  + 2 K2) Sin[θMc]^2 Sin[ϕMc]^2 + K2 Sin[θMc]^4 Sin[ϕMc]^4 + K1 + K2
```

Impose Chappert's condition on the static field $\left(\theta Hc = \frac{\pi}{2}\right)$

```
θHc = π/2;

F

1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 - H M (Cos[ϕHc] Cos[ϕMc] Sin[θMc] + Sin[θMc] Sin[ϕHc] Sin[ϕMc]) +
 K1 (1 - Sin[θMc]^2 Sin[ϕMc]^2) + K2 (1 - Sin[θMc]^2 Sin[ϕMc]^2)^2

F = -H M (Cos[ϕHc] Cos[ϕMc] Sin[θMc] + Sin[θMc] Sin[ϕHc] Sin[ϕMc]) +
  1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 - (K1  + 2 K2) Sin[θMc]^2 Sin[ϕMc]^2 + K2 Sin[θMc]^4 Sin[ϕMc]^4 + K1 + K2
```

Using the trigonometric identity

```
Cos[ϕHc - ϕMc] = Cos[ϕHc] Cos[ϕMc] + Sin[ϕHc] Sin[ϕMc];

F = -H M Sin[θMc] Cos[ϕHc - ϕMc] + 1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 -
  (K1  + 2 K2) Sin[θMc]^2 Sin[ϕMc]^2 + K2 Sin[θMc]^4 Sin[ϕMc]^4 + K1 + K2
```

This expression of the free energy is identical to Chappert's expression except for the constant terms K1 and K2. These terms do not have an effect on any of the results for any one of three reasons. 1. The constant terms K1 and K2 can be merged with the K0 term in the uniaxial anisotropy energy exponential series. 2. The terms are simply an offset to the already arbitrary energy reference point. 3. Further analysis requires differentiation of the free energy in which the constant terms K1 and K2 drop out.

Therefore without loss of generality we can express the free energy exactly as Chappert did.

```
F = -H M Sin[θMc] Cos[ϕHc - ϕMc] + 1/2 M^2 Nz Sin[θMc]^2 Sin[ϕMc]^2 -
  (K1  + 2 K2) Sin[θMc]^2 Sin[ϕMc]^2 + K2 Sin[θMc]^4 Sin[ϕMc]^4
```

Appendix C

Chappert Derivation of Resonant Frequency

This work starts with the free energy expression given in Chappert ' s work and follows the derivation process ending with an expression for the resonant frequency.

Notation :

Due to the way Mathematica works with subscripts the variables normally expressed with subscripts will simply be expressed by the following letter of the subscript. i.e.

θ_M = θM; ϕ_M = ϕM; θ_H = θH; ϕ_H = ϕH;

All variables are in the Chappert coordinate system.

Also terms such as FθM = F_{θ_M} and denote partial derivatives.

e.g. FϕMϕM = $F_{\phi_M \phi_M} = \frac{\partial^2 F}{\partial^2 \phi_M}$

ϕH =.; θH =.; ϕM =.; θM =.;

Free Energy in Chappert Coordinates

$$F = -M H \operatorname{Sin}[\theta M] \operatorname{Cos}[\phi H - \phi M] + \frac{1}{2} 4 \pi M^2 (\operatorname{Sin}[\theta M])^2 (\operatorname{Sin}[\phi M])^2 - (K1 + 2 K2) (\operatorname{Sin}[\theta M])^2 (\operatorname{Sin}[\phi M])^2 + K2 (\operatorname{Sin}[\theta M])^4 (\operatorname{Sin}[\phi M])^4;$$

Partial Derivatives

FθM = D[F, θM]

$$-H M \operatorname{Cos}[\theta M] \operatorname{Cos}[\phi H - \phi M] - 2 (K1 + 2 K2) \operatorname{Cos}[\theta M] \operatorname{Sin}[\theta M] \operatorname{Sin}[\phi M]^2 + 4 M^2 \pi \operatorname{Cos}[\theta M] \operatorname{Sin}[\theta M] \operatorname{Sin}[\phi M]^2 + 4 K2 \operatorname{Cos}[\theta M] \operatorname{Sin}[\theta M]^3 \operatorname{Sin}[\phi M]^4$$

FθMθM = D[F, {θM, 2}]

$$H M \operatorname{Cos}[\phi H - \phi M] \operatorname{Sin}[\theta M] - (K1 + 2 K2) \left(2 \operatorname{Cos}[\theta M]^2 - 2 \operatorname{Sin}[\theta M]^2\right) \operatorname{Sin}[\phi M]^2 + 2 M^2 \pi \left(2 \operatorname{Cos}[\theta M]^2 - 2 \operatorname{Sin}[\theta M]^2\right) \operatorname{Sin}[\phi M]^2 + K2 \left(12 \operatorname{Cos}[\theta M]^2 \operatorname{Sin}[\theta M]^2 - 4 \operatorname{Sin}[\theta M]^4\right) \operatorname{Sin}[\phi M]^4$$

FϕM = D[F, ϕM]

$$-H M \operatorname{Sin}[\theta M] \operatorname{Sin}[\phi H - \phi M] - 2 (K1 + 2 K2) \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M]^2 \operatorname{Sin}[\phi M] + 4 M^2 \pi \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M]^2 \operatorname{Sin}[\phi M] + 4 K2 \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M]^4 \operatorname{Sin}[\phi M]^3$$

FϕMϕM = D[F, {ϕM, 2}]

$$H M \operatorname{Cos}[\phi H - \phi M] \operatorname{Sin}[\theta M] - (K1 + 2 K2) \operatorname{Sin}[\theta M]^2 \left(2 \operatorname{Cos}[\phi M]^2 - 2 \operatorname{Sin}[\phi M]^2\right) + 2 M^2 \pi \operatorname{Sin}[\theta M]^2 \left(2 \operatorname{Cos}[\phi M]^2 - 2 \operatorname{Sin}[\phi M]^2\right) + K2 \operatorname{Sin}[\theta M]^4 \left(12 \operatorname{Cos}[\phi M]^2 \operatorname{Sin}[\phi M]^2 - 4 \operatorname{Sin}[\phi M]^4\right)$$

FϕMθM = D[FϕM, θM]

$$-H M \operatorname{Cos}[\theta M] \operatorname{Sin}[\phi H - \phi M] - 4 (K1 + 2 K2) \operatorname{Cos}[\theta M] \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M] \operatorname{Sin}[\phi M] + 8 M^2 \pi \operatorname{Cos}[\theta M] \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M] \operatorname{Sin}[\phi M] + 16 K2 \operatorname{Cos}[\theta M] \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M]^3 \operatorname{Sin}[\phi M]^3$$

FθMϕM = D[FθM, ϕM]

$$-H M \operatorname{Cos}[\theta M] \operatorname{Sin}[\phi H - \phi M] - 4 (K1 + 2 K2) \operatorname{Cos}[\theta M] \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M] \operatorname{Sin}[\phi M] + 8 M^2 \pi \operatorname{Cos}[\theta M] \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M] \operatorname{Sin}[\phi M] + 16 K2 \operatorname{Cos}[\theta M] \operatorname{Cos}[\phi M] \operatorname{Sin}[\theta M]^3 \operatorname{Sin}[\phi M]^3$$

Smit and Beljers result

$$\left(\frac{\omega}{\gamma}\right)^2 = \frac{1}{M^2\,\mathrm{Sin}[\theta M]^2}\left(F\theta M\theta M\, F\phi M\phi M - (F\theta M\phi M)^2\right)$$

$$g = \frac{1}{M^2\,\mathrm{Sin}[\theta M]^2}\left(F\theta M\theta M\, F\phi M\phi M - (F\theta M\phi M)^2\right)$$

$$\frac{1}{M^2}\mathrm{Csc}[\theta M]^2 \Big(-\big(-H\,M\,\mathrm{Cos}[\theta M]\,\mathrm{Sin}[\phi H-\phi M] - 4\,(K1+2\,K2)\,\mathrm{Cos}[\theta M]\,\mathrm{Cos}[\phi M]\,\mathrm{Sin}[\theta M]\,\mathrm{Sin}[\phi M] + 8\,M^2\pi\,\mathrm{Cos}[\theta M]\,\mathrm{Cos}[\phi M]\,\mathrm{Sin}[\theta M]\,\mathrm{Sin}[\phi M] + 16\,K2\,\mathrm{Cos}[\theta M]\,\mathrm{Cos}[\phi M]\,\mathrm{Sin}[\theta M]^3\,\mathrm{Sin}[\phi M]^3\big)^2 + \big(H\,M\,\mathrm{Cos}[\phi H-\phi M]\,\mathrm{Sin}[\theta M] - (K1+2\,K2)\left(2\,\mathrm{Cos}[\theta M]^2 - 2\,\mathrm{Sin}[\theta M]^2\right)\mathrm{Sin}[\phi M]^2 + 2\,M^2\pi\left(2\,\mathrm{Cos}[\theta M]^2 - 2\,\mathrm{Sin}[\theta M]^2\right)\mathrm{Sin}[\phi M]^2 + K2\left(12\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2 - 4\,\mathrm{Sin}[\theta M]^4\right)\mathrm{Sin}[\phi M]^4\big) \big(H\,M\,\mathrm{Cos}[\phi H-\phi M]\,\mathrm{Sin}[\theta M] - (K1+2\,K2)\,\mathrm{Sin}[\theta M]^2\left(2\,\mathrm{Cos}[\phi M]^2 - 2\,\mathrm{Sin}[\phi M]^2\right) + 2\,M^2\pi\,\mathrm{Sin}[\theta M]^2\left(2\,\mathrm{Cos}[\phi M]^2 - 2\,\mathrm{Sin}[\phi M]^2\right) + K2\,\mathrm{Sin}[\theta M]^4\left(12\,\mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\phi M]^2 - 4\,\mathrm{Sin}[\phi M]^4\right)\big)\Big)$$

Simplify[g]

$$\frac{1}{M^2}\mathrm{Csc}[\theta M]^2 \Big(-\mathrm{Cos}[\theta M]^2\left(H\,M\,\mathrm{Sin}[\phi H-\phi M] + 4\,\mathrm{Cos}[\phi M]\,\mathrm{Sin}[\theta M]\,\mathrm{Sin}[\phi M]\left(K1+2\,K2-2\,M^2\pi - 4\,K2\,\mathrm{Sin}[\theta M]^2\,\mathrm{Sin}[\phi M]^2\right)\right)^2 + \mathrm{Sin}[\theta M]\left(H\,M\,\mathrm{Cos}[\phi H-\phi M] - 2\,(K1+2\,K2)\,\mathrm{Cos}[2\,\phi M]\,\mathrm{Sin}[\theta M] + 4\,M^2\pi\,\mathrm{Cos}[2\,\phi M]\,\mathrm{Sin}[\theta M] + 4\,K2\,(1+2\,\mathrm{Cos}[2\,\phi M])\,\mathrm{Sin}[\theta M]^3\,\mathrm{Sin}[\phi M]^2\right)\left(H\,M\,\mathrm{Cos}[\phi H-\phi M]\,\mathrm{Sin}[\theta M] + 2\,\mathrm{Sin}[\phi M]^2\left(2\,K2\,\mathrm{Sin}[\theta M]^2\,\mathrm{Sin}[\phi M]^2 - \mathrm{Cos}[2\,\theta M]\left(K1+2\,K2-2\,M^2\pi-4\,K2\,\mathrm{Sin}[\theta M]^2\,\mathrm{Sin}[\phi M]^2\right)\right)\right)\Big)$$

Set $\theta M = \theta Meq = \pi/2$;

$$\theta M = \pi/2;$$

$$g = \frac{1}{M^2\,\mathrm{Sin}[\theta M]^2}\left(F\theta M\theta M\, F\phi M\phi M - (F\theta M\phi M)^2\right)$$

$$\frac{1}{M^2}\left(H\,M\,\mathrm{Cos}[\phi H-\phi M] + 2\,(K1+2\,K2)\,\mathrm{Sin}[\phi M]^2 - 4\,M^2\pi\,\mathrm{Sin}[\phi M]^2 - 4\,K2\,\mathrm{Sin}[\phi M]^4\right)\left(H\,M\,\mathrm{Cos}[\phi H-\phi M] - (K1+2\,K2)\left(2\,\mathrm{Cos}[\phi M]^2 - 2\,\mathrm{Sin}[\phi M]^2\right) + 2\,M^2\pi\left(2\,\mathrm{Cos}[\phi M]^2 - 2\,\mathrm{Sin}[\phi M]^2\right) + K2\left(12\,\mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\phi M]^2 - 4\,\mathrm{Sin}[\phi M]^4\right)\right)$$

Simplify[g]

$$\frac{1}{M^2}\left(H\,M\,\mathrm{Cos}[\phi H-\phi M] - 2\left(\left(K1+K2-2\,M^2\pi\right)\mathrm{Cos}[2\,\phi M] + K2\,\mathrm{Cos}[4\,\phi M]\right)\right)\left(H\,M\,\mathrm{Cos}[\phi H-\phi M] + 2\left(K1+K2-2\,M^2\pi + K2\,\mathrm{Cos}[2\,\phi M]\right)\mathrm{Sin}[\phi M]^2\right)$$

$$\left(H\,\mathrm{Cos}[\phi H-\phi M] - \left(\left(\frac{2\,K1}{M}+\frac{2\,K2}{M}-4\,M\pi\right)\mathrm{Cos}[2\,\phi M] + \frac{2\,K2}{M}\mathrm{Cos}[4\,\phi M]\right)\right)\left(H\,\mathrm{Cos}[\phi H-\phi M] + \left(\frac{2\,K1}{M}+\frac{2\,K2}{M}-4\,M\pi+\frac{2\,K2}{M}\mathrm{Cos}[2\,\phi M]\right)\mathrm{Sin}[\phi M]^2\right);$$

Also Note :

$$HA1 = 2\,K1/M$$
$$HA2 = 4\,K2/M$$
$$HA = HA1 + HA2$$

Separate into H1 and H2 :

H1

$$H1 = \left(H\,Cos[\phi H - \phi M] + \left(\frac{2\,K1}{M} + \frac{2\,K2}{M} - 4\,M\,\pi + \frac{2\,K2}{M}\,Cos[2\,\phi M]\right) Sin[\phi M]^2\right)$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - \left(4\,M\,\pi - \left(\frac{2\,K1}{M} + \frac{2\,K2}{M}\right) - \frac{2\,K2}{M}\,Cos[2\,\phi M]\right) Sin[\phi M]^2\right)$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - \left(4\,\pi\,M - \left(\frac{2\,K1}{M} + \frac{2\,K2}{M}\right)\right) Sin[\phi M]^2 + \frac{2\,K2}{M}\,Cos[2\,\phi M]\,Sin[\phi M]^2\right)$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - \left(4\,\pi\,M - \left(\frac{2\,K1}{M} + \frac{4\,K2}{M}\right)\right) Sin[\phi M]^2 - \frac{2\,K2}{M}\,Sin[\phi M]^2 + \frac{2\,K2}{M}\,Cos[2\,\phi M]\,Sin[\phi M]^2\right)$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - \left(4\,\pi\,M - \left(\frac{2\,K1}{M} + \frac{4\,K2}{M}\right)\right) Sin[\phi M]^2 - \frac{2\,K2}{M}\,Sin[\phi M]^2\,\left(1 - Cos[2\,\phi M]\,\right)\right)$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - (4\,\pi\,M - HA)\,Sin[\phi M]^2 - \frac{2\,K2}{M}\,Sin[\phi M]^2\,\left(1 - Cos[2\,\phi M]\,\right)\right)$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - (4\,\pi\,M - HA)\,Sin[\phi M]^2 - \frac{4\,K2}{M}\,Sin[\phi M]^2\,\frac{1}{2}\,\left(1 - Cos[2\,\phi M]\,\right)\right)$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - (4\,\pi\,M - HA)\,Sin[\phi M]^2 - HA2\,Sin[\phi M]^2\,\frac{1}{2}\,\left(1 - Cos[2\,\phi M]\right)\right)$$

$$TrigFactor\left[Sin[\phi M]^2\,\frac{1}{2}\,\left(1 - Cos[2\,\phi M]\,\right)\right]$$

$$Sin[\phi M]^4$$

$$H1 = \left(H\,Cos[\phi H - \phi M] - (4\,\pi\,M - HA)\,Sin[\phi M]^2 - HA2\,Sin[\phi M]^4\right)$$

From Chappert text :

$$H1 = \left(H\,Cos[\phi H - \phi M] - (4\,\pi\,M - HA)\,Sin[\phi M]^2 - HA2\,Sin[\phi M]^4\right)$$

H1 Agrees

H2

$$H2 = \left(H\,Cos[\phi H - \phi M] - \left(\left(\frac{2\,K1}{M} + \frac{2\,K2}{M} - 4\,M\,\pi\right) Cos[2\,\phi M] + \frac{2\,K2}{M}\,Cos[4\,\phi M]\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + \left(\left(4\,M\,\pi - \left(\frac{2\,K1}{M} + \frac{2\,K2}{M}\right)\right) Cos[2\,\phi M] - \frac{2\,K2}{M}\,Cos[4\,\phi M]\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + \left(\left(4\,\pi\,M - \left(\frac{2\,K1}{M} + \frac{2\,K2}{M}\right)\right) Cos[2\,\phi M] - \frac{2\,K2}{M}\,Cos[4\,\phi M]\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + \left(\left(4\,\pi\,M - \left(\frac{2\,K1}{M} + \frac{4\,K2}{M}\right)\right) Cos[2\,\phi M] + \frac{2\,K2}{M}\,Cos[2\,\phi M] - \frac{2\,K2}{M}\,Cos[4\,\phi M]\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + \left(\left(4\,\pi\,M - \left(\frac{2\,K1}{M} + \frac{4\,K2}{M}\right)\right) Cos[2\,\phi M] + \frac{2\,K2}{M}\,\left(Cos[2\,\phi M] - Cos[4\,\phi M]\right)\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + (4\,\pi\,M - HA)\,Cos[2\,\phi M] + \frac{2\,K2}{M}\,\left(Cos[2\,\phi M] - Cos[4\,\phi M]\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + (4\,\pi\,M - HA)\,Cos[2\,\phi M] + \frac{4\,K2}{M}\,\left(\frac{1}{2}\,Cos[2\,\phi M] - \frac{1}{2}\,Cos[4\,\phi M]\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + (4\,\pi\,M - HA)\,Cos[2\,\phi M] + HA2\,\left(\frac{1}{2}\,Cos[2\,\phi M] - \frac{1}{2}\,Cos[4\,\phi M]\right)\right)$$

$Cos[2\,\phi M] = 1 - 2\,Sin[\phi M]^2$

$Cos[4\,\phi M]$:

TrigExpand[Cos[4 ϕM]]

$Cos[\phi M]^4 - 6\,Cos[\phi M]^2\,Sin[\phi M]^2 + Sin[\phi M]^4$

$Cos[\phi M]^4$:

$$\begin{aligned}
Cos[\phi M]^4 &= \left(Cos[\phi M]^2\right)^2 \\
&= \left(1 - Sin[\phi M]^2\right)^2 \\
&= 1 - 2\,Sin[\phi M]^2 + Sin[\phi M]^4 \\
&= Cos[2\,\phi M] + Sin[\phi M]^4
\end{aligned}$$

$Cos[4\,\phi M]$

$$\begin{aligned}
&= Cos[\phi M]^4 - 6\,Cos[\phi M]^2\,Sin[\phi M]^2 + Sin[\phi M]^4 \\
&= Cos[2\,\phi M] + Sin[\phi M]^4 - 6\,Cos[\phi M]^2\,Sin[\phi M]^2 + Sin[\phi M]^4 \\
&= Cos[2\,\phi M] - 6\,Cos[\phi M]^2\,Sin[\phi M]^2 + 2\,Sin[\phi M]^4
\end{aligned}$$

$$H2 = \Bigg(H\,Cos[\phi H - \phi M] + (4\,\pi\,M - HA)\,Cos[2\,\phi M] + HA2\,\left(\frac{1}{2}\,Cos[2\,\phi M] - \frac{1}{2}\,\left(Cos[2\,\phi M] - 6\,Cos[\phi M]^2\,Sin[\phi M]^2 + 2\,Sin[\phi M]^4\right)\right)\Bigg)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + (4\,\pi\,M - HA)\,Cos[2\,\phi M] + HA2\,\left(-\frac{1}{2}\,\left(-6\,Cos[\phi M]^2\,Sin[\phi M]^2 + 2\,Sin[\phi M]^4\right)\right)\right)$$

$$H2 = \left(H\,Cos[\phi H - \phi M] + (4\,\pi\,M - HA)\,Cos[2\,\phi M] + HA2\,\left(\left(3\,Cos[\phi M]^2\,Sin[\phi M]^2 - Sin[\phi M]^4\right)\right)\right)$$

From Chappert text :

$$H2 = \left(H\,Cos[\phi H - \phi M] + (4\,\pi\,M - HA)\,Cos[2\,\phi M] + HA2\,\left(3\,Sin[\phi M]^2\,Cos[\phi M]^2 - Sin[\phi M]^4\right)\right)$$

H2 Agrees

Appendix D

Chappert ' s virtual fields H1 H2 Coordinate Conversion to Vonsovskii Coordinates

This work starts with the virtual fields expression given in Chappert ' s work for the resonant frequency and follows the conversion process ending with a matching expression for both virtual fields given in Okada ' s work assuming Okada ' s conditions.

Notation :

Due to the way Mathematica works with subscripts the variables normally expressed with subscripts will simply be expressed by the following letter of the subscript. i.e.

$\theta_M = \theta M$; $\phi_M = \phi M$; $\theta_H = \theta H$; $\phi_H = \phi H$;

Also variables followed by a c denote that coordinate is in the Chappert coordinate system

Note : Okada ' s imposed condition $\phi M = \phi H$

($\phi M = \phi H$ was found by setting Zeeman energies equal in both the general Vonsovskii expression and the Okada expression)

Vonsovskii :

$F^{V}_{Zee} = -M\,H\,(\mathrm{Cos}[\phi H - \phi M]\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + \mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M])$

Okada :

$F^{O}_{Zee} = -M\,H\,(\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + \mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M])$

For $F^{V}_{Zee} = F^{O}_{Zee}$

$\mathrm{Cos}[\phi H - \phi M]$ must $= 1$ generally

Therefore $\phi M = \phi H$

$\phi H = .;\ \theta H = .;\ \phi M = .;\ \theta M = .;$

$\phi Hc = .;\ \theta Hc = .;\ \phi Mc = .;\ \theta Mc = .;$

$$\left(\frac{\omega}{\gamma}\right)^2 = H1\,H2;$$

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(H\,\mathrm{Cos}[\phi Hc - \phi Mc] + (4\,\pi\,M - HA)\,\mathrm{Cos}[2\,\phi Mc] + HA2\,\left(3\,\mathrm{Sin}[\phi Mc]^2\,\mathrm{Cos}[\phi Mc]^2 - \mathrm{Sin}[\phi Mc]^4\right)\right)$$
$$\left(H\,\mathrm{Cos}[\phi Hc - \phi Mc] - (4\,\pi\,M - HA)\,\mathrm{Sin}[\phi Mc]^2 - HA2\,\mathrm{Sin}[\phi Mc]^4\right)$$

$$HA1 = 2\,K1 / M$$
$$HA2 = 4\,K2 / M$$
$$HA = HA1 + HA2$$

$$\mathrm{Cos}[\theta Mc] = \mathrm{Cos}[\phi M]\,\mathrm{Sin}[\theta M];$$
$$\mathrm{Sin}[\theta Mc] = \sqrt{\left(1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2\right)};$$
$$\mathrm{Cos}[\phi Mc] = (\mathrm{Sin}[\phi M]\,\mathrm{Sin}[\theta M]) / \left(\sqrt{\left(1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2\right)}\right);$$
$$\mathrm{Sin}[\phi Mc] = \mathrm{Cos}[\theta M] / \left(\sqrt{\left(1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2\right)}\right);$$

$$\mathrm{Cos}[\phi Hc - \phi Mc] = \mathrm{Cos}[\phi Hc]\,\mathrm{Cos}[\phi Mc] + \mathrm{Sin}[\phi Hc]\,\mathrm{Sin}[\phi Mc];$$

$$\mathrm{Cos}[\phi Hc] = \mathrm{Sin}[\theta H];$$
$$\mathrm{Sin}[\phi Hc] = \mathrm{Cos}[\theta H];$$

H1

```
--------------------------------------------------------------
```

```
H1 = (H Cos[ϕHc - ϕMc] - (4 π M - HA) Sin[ϕMc]^2 - HA2 Sin[ϕMc]^4);
H1 = (H Cos[ϕHc - ϕMc] - (4 π M - (HA1 + HA2)) Sin[ϕMc]^2 - (4 K2 / M) Sin[ϕMc]^4);
H1 = (H Cos[ϕHc - ϕMc] - (4 π M - ((2 K1 / M) + (4 K2 / M))) Sin[ϕMc]^2 - (4 K2 / M) Sin[ϕMc]^4);
H1 = (H (Cos[ϕHc] Cos[ϕMc] + Sin[ϕHc] Sin[ϕMc]) -
    (4 π M - ((2 K1 / M) + (4 K2 / M))) Sin[ϕMc]^2 - (4 K2 / M) Sin[ϕMc]^4);
H1 = (H (Sin[θH] ((Sin[ϕM] Sin[θM]) / (√(1 - Cos[ϕM]^2 Sin[θM]^2))) +
        Cos[θH] (Cos[θM] / (√(1 - Cos[ϕM]^2 Sin[θM]^2)))) -
    (4 π M - ((2 K1 / M) + (4 K2 / M))) (Cos[θM] / (√(1 - Cos[ϕM]^2 Sin[θM]^2)))^2 -
    (4 K2 / M) (Cos[θM] / (√(1 - Cos[ϕM]^2 Sin[θM]^2)))^4);
```

```
Simplify[H1]
```

```
-((4 K2 Cos[θM]^4) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)^2)) -
 (2 (K1 + 2 K2 - 2 M^2 π) Cos[θM]^2) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)) +
 (H (Cos[θH] Cos[θM] + Sin[θH] Sin[θM] Sin[ϕM])) / (√(1 - Cos[ϕM]^2 Sin[θM]^2))
```

```
Collect[-((4 K2 Cos[θM]^4) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)^2)) -
   (2 (K1 + 2 K2 - 2 M^2 π) Cos[θM]^2) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)) +
   (H (Cos[θH] Cos[θM] + Sin[θH] Sin[θM] Sin[ϕM])) / (√(1 - Cos[ϕM]^2 Sin[θM]^2)), K1];

Collect[-((4 K2 Cos[θM]^4) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)^2)) - (2 K1 Cos[θM]^2) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)) -
   (4 K2 Cos[θM]^2) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)) + (4 M π Cos[θM]^2) / (-1 + Cos[ϕM]^2 Sin[θM]^2) +
   (H (Cos[θH] Cos[θM] + Sin[θH] Sin[θM] Sin[ϕM])) / (√(1 - Cos[ϕM]^2 Sin[θM]^2)), K2];

H1 = -((2 K1 Cos[θM]^2) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2))) + (4 M π Cos[θM]^2) / (-1 + Cos[ϕM]^2 Sin[θM]^2) +
  K2 (-((4 Cos[θM]^4) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2)^2)) - (4 Cos[θM]^2) / (M (-1 + Cos[ϕM]^2 Sin[θM]^2))) +
  (H (Cos[θH] Cos[θM] + Sin[θH] Sin[θM] Sin[ϕM])) / (√(1 - Cos[ϕM]^2 Sin[θM]^2))
```

Impose conditions implied by Okada

```
ϕM = ϕH = π / 2;
```

$$\texttt{H1} = -\left(\left(2\,\texttt{K1}\,\texttt{Cos}[\theta M]^2\right) / \left(M\,(-1)\right)\right) + \frac{1}{-1}\,4\,M\,\pi\,\texttt{Cos}[\theta M]^2 + \texttt{K2}\left(-\left(\left(4\,\texttt{Cos}[\theta M]^4\right)/\left(M\,(-1)^2\right)\right) - \left(4\,\texttt{Cos}[\theta M]^2\right)/\left(M\,(-1)\right)\right) + \left(H\,(\texttt{Cos}[\theta H]\,\texttt{Cos}[\theta M] + \texttt{Sin}[\theta H]\,\texttt{Sin}[\theta M]\,)\right)/\left(\sqrt{1}\right);$$

```
TrigFactor[H1]
```

$$-\frac{1}{2\,M}\left(-2\,\texttt{K1} - \texttt{K2} + 4\,M^2\,\pi - 2\,H\,M\,\texttt{Cos}[\theta H - \theta M] + \left(-2\,\texttt{K1} + 4\,M^2\,\pi\right)\texttt{Cos}[2\,\theta M] + \texttt{K2}\,\texttt{Cos}[4\,\theta M]\right)$$

$$\texttt{H1} = -\left(\frac{-2\,\texttt{K1}}{2\,M} - \frac{\texttt{K2}}{2\,M} + \frac{4\,M^2\,\pi}{2\,M} - \left(2\,H\,M\,\texttt{Cos}[\theta H - \theta M]\right)/\left(2\,M\right) + \left(-\frac{2\,\texttt{K1}}{2\,M} + \frac{4\,M^2\,\pi}{2\,M}\right)\texttt{Cos}[2\,\theta M] + \frac{\texttt{K2}}{2\,M}\,\texttt{Cos}[4\,\theta M]\right);$$

$$\texttt{H1} = \frac{2\,\texttt{K1}}{2\,M} + \frac{\texttt{K2}}{2\,M} - \frac{4\,M^2\,\pi}{2\,M} + \left(2\,H\,M\,\texttt{Cos}[\theta H - \theta M]\right)/\left(2\,M\right) - \left(-\frac{2\,\texttt{K1}}{2\,M} + \frac{4\,M^2\,\pi}{2\,M}\right)\texttt{Cos}[2\,\theta M] - \frac{\texttt{K2}}{2\,M}\,\texttt{Cos}[4\,\theta M];$$

$$\texttt{H1} = \frac{\texttt{K1}}{M} + \frac{\texttt{K2}}{2\,M} - \frac{4\,\pi\,M}{2} + H\,\texttt{Cos}[\theta H - \theta M] - \left(-\frac{\texttt{K1}}{M} + \frac{4\,\pi\,M}{2}\right)\texttt{Cos}[2\,\theta M] - \frac{\texttt{K2}}{2\,M}\,\texttt{Cos}[4\,\theta M];$$

$$\texttt{H1} = H\,\texttt{Cos}[\theta H - \theta M] - \left(-\frac{\texttt{K1}}{M} + \frac{4\,\pi\,M}{2}\right)\texttt{Cos}[2\,\theta M] - \frac{\texttt{K2}}{2\,M}\,\texttt{Cos}[4\,\theta M] + \frac{\texttt{K1}}{M} + \frac{\texttt{K2}}{2\,M} - \frac{4\,\pi\,M}{2};$$

$\text{Cos}[2\,\theta M] = 1 - 2\,\text{Sin}[\theta M]^2$

$\text{Cos}[4\,\theta M]$:

$\text{TrigExpand}[\text{Cos}[4\,\theta M]]$

$\text{Cos}[\theta M]^4 - 6\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2 + \text{Sin}[\theta M]^4$

$\text{Cos}[\theta M]^4$:

$$
\begin{aligned}
\text{Cos}[\theta M]^4 &= \left(\text{Cos}[\theta M]^2\right)^2 \\
&= \left(1 - \text{Sin}[\theta M]^2\right)^2 \\
&= 1 - 2\,\text{Sin}[\theta M]^2 + \text{Sin}[\theta M]^4 \\
&= \text{Cos}[2\,\theta M] + \text{Sin}[\theta M]^4
\end{aligned}
$$

$\text{Cos}[4\,\theta M]$:

$$
\begin{aligned}
&= \text{Cos}[\theta M]^4 - 6\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2 + \text{Sin}[\theta M]^4 \\
&= \text{Cos}[2\,\theta M] + \text{Sin}[\theta M]^4 - 6\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2 + \text{Sin}[\theta M]^4 \\
&= \text{Cos}[2\,\theta M] - 6\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2 + 2\,\text{Sin}[\theta M]^4
\end{aligned}
$$

$$\text{Cos}[4\,\theta M] = \left(1 - 2\,\text{Sin}[\theta M]^2\right) - 6\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2 + 2\,\text{Sin}[\theta M]^4$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] - \left(-\frac{\text{K1}}{\text{M}} + \frac{4\,\pi\,\text{M}}{2}\right)\left(1 - 2\,\text{Sin}[\theta M]^2\right) - \frac{\text{K2}}{2\,\text{M}}\left(\left(1 - 2\,\text{Sin}[\theta M]^2\right) - 6\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2 + 2\,\text{Sin}[\theta M]^4\right) + \frac{\text{K1}}{\text{M}} + \frac{\text{K2}}{2\,\text{M}} - \frac{4\,\pi\,\text{M}}{2};$$

$\text{Simplify}[$
$\text{H1}]$

$$\frac{1}{\text{M}}\left(\text{H}\,\text{M}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\text{K1} + \text{K2} - 2\,\text{M}^2\,\pi - \text{K2}\,\text{Cos}[2\,\theta M]\right)\right)$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\frac{\text{K1}}{\text{M}} + \frac{\text{K2}}{\text{M}} - 2\,\text{M}\,\pi - \frac{\text{K2}}{\text{M}}\,\text{Cos}[2\,\theta M]\right);$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\frac{\text{K1}}{\text{M}} + \frac{\text{K2}}{\text{M}} - 2\,\text{M}\,\pi - \frac{\text{K2}}{\text{M}}\left(1 - 2\,\text{Sin}[\theta M]^2\right)\right);$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\frac{\text{K1}}{\text{M}} + \frac{\text{K2}}{\text{M}} - 2\,\text{M}\,\pi - \frac{\text{K2}}{\text{M}}\left(1 - 2\left(1 - \text{Cos}[\theta M]^2\right)\right)\right);$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\frac{\text{K1}}{\text{M}} + \frac{\text{K2}}{\text{M}} - 2\,\text{M}\,\pi - \frac{\text{K2}}{\text{M}} + \frac{2\,\text{K2}}{\text{M}}\left(1 - \text{Cos}[\theta M]^2\right)\right);$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\frac{\text{K1}}{\text{M}} - \frac{4\,\pi\,\text{M}}{2} + \frac{2\,\text{K2}}{\text{M}}\left(1 - \text{Cos}[\theta M]^2\right)\right);$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\frac{\text{K1}}{\text{M}} - \frac{4\,\pi\,\text{M}}{2} + \frac{2\,\text{K2}}{\text{M}} - \frac{2\,\text{K2}}{\text{M}}\,\text{Cos}[\theta M]^2\right);$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + 2\,\text{Cos}[\theta M]^2\left(\frac{\text{K1}}{\text{M}} - \frac{4\,\pi\,\text{M}}{2} + \frac{2\,\text{K2}}{\text{M}}\right) - \frac{2\,\text{K2}}{\text{M}}\,\text{Cos}[\theta M]^4;$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + \left(-4\,\pi\,\text{M} + \frac{2\,\text{K1}}{\text{M}} + \frac{4\,\text{K2}}{\text{M}}\right)\text{Cos}[\theta M]^2 - \frac{4\,\text{K2}}{\text{M}}\,\text{Cos}[\theta M]^4;$$

$$\text{HK1eff} = -4\,\pi\,\text{M} + \frac{2\,\text{K1}}{\text{M}} + \frac{4\,\text{K2}}{\text{M}};$$

$$\text{HK2} = \frac{4\,\text{K2}}{\text{M}};$$

$$\text{H1} = \text{H}\,\text{Cos}[\theta H - \theta M] + \text{HK1eff}\,\text{Cos}[\theta M]^2 - \text{HK2}\,\text{Cos}[\theta M]^4;$$

$\mathrm{Cos}[\theta Mc] = \mathrm{Cos}[\phi M]\,\mathrm{Sin}[\theta M];$

$\mathrm{Sin}[\theta Mc] = \sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2};$

$\mathrm{Cos}[\phi Mc] = \dfrac{\mathrm{Sin}[\phi M]\,\mathrm{Sin}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}};$

$\mathrm{Sin}[\phi Mc] = \dfrac{\mathrm{Cos}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}};$

$\mathrm{Cos}[\phi Hc - \phi Mc] = \mathrm{Cos}[\phi Hc]\,\mathrm{Cos}[\phi Mc] + \mathrm{Sin}[\phi Hc]\,\mathrm{Sin}[\phi Mc];$

$\mathrm{Cos}[\phi Hc] = \mathrm{Sin}[\theta H];$

$\mathrm{Sin}[\phi Hc] = \mathrm{Cos}[\theta H];$

H2

$H2 = \left(H\,\mathrm{Cos}[\phi Hc - \phi Mc] + (4\pi M - HA)\,\mathrm{Cos}[2\,\phi Mc] + HA2\,\left(3\,\mathrm{Sin}[\phi Mc]^2\,\mathrm{Cos}[\phi Mc]^2 - \mathrm{Sin}[\phi Mc]^4\right)\right);$

$H2 = \left(H\,\mathrm{Cos}[\phi Hc - \phi Mc] + \left(4\pi M - \left(HA1 + HA2\right)\right)\mathrm{Cos}[2\,\phi Mc] + HA2\,\left(3\,\mathrm{Sin}[\phi Mc]^2\,\mathrm{Cos}[\phi Mc]^2 - \mathrm{Sin}[\phi Mc]^4\right)\right);$

$H2 = \left(H\,\mathrm{Cos}[\phi Hc - \phi Mc] + \left(4\pi M - \left(2\,K1/M + 4\,K2/M\right)\right)\mathrm{Cos}[2\,\phi Mc] + 4\,K2/M\,\left(3\,\mathrm{Sin}[\phi Mc]^2\,\mathrm{Cos}[\phi Mc]^2 - \mathrm{Sin}[\phi Mc]^4\right)\right);$

$$H2 = \Big(H\,\mathrm{Cos}[\phi Hc - \phi Mc] + \left(4\pi M - \left(2\,K1/M + 4\,K2/M\right)\right)\left(1 - 2\,\mathrm{Sin}[\phi Mc]^2\right) + 4\,K2/M\,\left(3\,\mathrm{Sin}[\phi Mc]^2\,\mathrm{Cos}[\phi Mc]^2 - \mathrm{Sin}[\phi Mc]^4\right)\Big);$$

$$H2 = \Big(H\,(\mathrm{Cos}[\phi Hc]\,\mathrm{Cos}[\phi Mc] + \mathrm{Sin}[\phi Hc]\,\mathrm{Sin}[\phi Mc]) + \left(4\pi M - \left(2\,K1/M + 4\,K2/M\right)\right)\left(1 - 2\,\mathrm{Sin}[\phi Mc]^2\right) + 4\,K2/M\,\left(3\,\mathrm{Sin}[\phi Mc]^2\,\mathrm{Cos}[\phi Mc]^2 - \mathrm{Sin}[\phi Mc]^4\right)\Big);$$

$$H2 = \left(H\left(\mathrm{Sin}[\theta H]\,\frac{\mathrm{Sin}[\phi M]\,\mathrm{Sin}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}} + \mathrm{Cos}[\theta H]\,\frac{\mathrm{Cos}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}}\right) + \left(4\pi M - \left(2\,K1/M + 4\,K2/M\right)\right)\left(1 - 2\left(\frac{\mathrm{Cos}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}}\right)^2\right) + 4\,K2/M\left(3\left(\frac{\mathrm{Cos}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}}\right)^2\left(\frac{\mathrm{Sin}[\phi M]\,\mathrm{Sin}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}}\right)^2 - \left(\frac{\mathrm{Cos}[\theta M]}{\sqrt{1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2}}\right)^4\right)\right);$$

Simplify[H2]

$$-\frac{1}{M}2\,\left(K1 + 2\,K2 - 2\,M^2\,\pi\right)\left(1 + \left(2\,\mathrm{Cos}[\theta M]^2\right)/\left(-1 + \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2\right)\right) + (H\,(\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] + \mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M]\,\mathrm{Sin}[\phi M]))/\left(\sqrt{\left(1 - \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2\right)}\right) - \left(4\,K2\,\left(\mathrm{Cos}[\theta M]^4 - 3\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\,\mathrm{Sin}[\phi M]^2\right)\right)/\left(M\,\left(-1 + \mathrm{Cos}[\phi M]^2\,\mathrm{Sin}[\theta M]^2\right)^2\right)$$

Impose conditions implied by Okada

$\phi M = \phi H = \pi/2;$

$$H2 = -\frac{1}{M}2\,\left(K1 + 2\,K2 - 2\,M^2\,\pi\right)\left(1 + \frac{1}{-1}2\,\mathrm{Cos}[\theta M]^2\right) + (H\,(\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] + \mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M]\,))/\left(\sqrt{1}\right) - \left(4\,K2\,\left(\mathrm{Cos}[\theta M]^4 - 3\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\,\right)\right)/\left(M\,(-1)^2\right);$$

TrigFactor[H2]

$$-\frac{1}{M}\left(-H\,M\,\mathrm{Cos}[\theta H - \theta M] + \left(-2\,K1 - 2\,K2 + 4\,M^2\,\pi\right)\mathrm{Cos}[2\,\theta M] + 2\,K2\,\mathrm{Cos}[4\,\theta M]\right)$$

$$H2 = -\frac{1}{M}\left(-H M \mathrm{Cos}[\theta H - \theta M] + \left(-2\,K1 - 2\,K2 + 4\,M^2\,\pi\right)\mathrm{Cos}[2\,\theta M] + 2\,K2\,\mathrm{Cos}[4\,\theta M]\right);$$

$$H2 = -\left(-H\ \mathrm{Cos}[\theta H - \theta M] + \left(-\frac{2\,K1}{M} - \frac{2\,K2}{M} + 4\,\pi\,M\right)\mathrm{Cos}[2\,\theta M] + \frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right);$$

$$H2 = H\ \mathrm{Cos}[\theta H - \theta M] - \left(4\,\pi\,M - \frac{2\,K1}{M} - \frac{2\,K2}{M}\right)\mathrm{Cos}[2\,\theta M] - \frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M];$$

$$H2 = H\ \mathrm{Cos}[\theta H - \theta M] + \left(-4\,\pi\,M + \frac{2\,K1}{M} + \frac{2\,K2}{M}\right)\mathrm{Cos}[2\,\theta M] - \frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M];$$

$$H2 = H\ \mathrm{Cos}[\theta H - \theta M] + \left(-4\,\pi\,M + \frac{2\,K1}{M} + \frac{4\,K2}{M}\right)\mathrm{Cos}[2\,\theta M] - \frac{2\,K2}{M}\mathrm{Cos}[2\,\theta M] - \frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M];$$

$$H2 = H\ \mathrm{Cos}[\theta H - \theta M] + \left(-4\,\pi\,M + \frac{2\,K1}{M} + \frac{4\,K2}{M}\right)\mathrm{Cos}[2\,\theta M] - \frac{2\,K2}{M}\left(\mathrm{Cos}[2\,\theta M] + \mathrm{Cos}[4\,\theta M]\right);$$

$$HK1eff = -4\,\pi\,M + \frac{2\,K1}{M} + \frac{4\,K2}{M};$$

$$HK2 = \frac{4\,K2}{M};$$

$$H2 = H\ \mathrm{Cos}[\theta H - \theta M] + HK1eff\,\mathrm{Cos}[2\,\theta M] - \frac{HK2}{2}\left(\mathrm{Cos}[2\,\theta M] + \mathrm{Cos}[4\,\theta M]\right);$$

Appendix E

For VCMA acting in the perpendicular *direction* : Kx = Ky = 0

$$\left(\frac{\omega}{\gamma}\right)^2 =$$

$$\Big(\text{Hx Sin}[\theta M]\,\text{Cos}[\phi M] + \text{Hy Sin}[\theta M]\,\text{Sin}[\phi M] + \text{Hz Cos}[\theta M] + \left(-M\left(\text{Nz} + \frac{\text{Kz}}{M^2}\right)\right)(\text{Cos}[\theta M]^2 - \text{Sin}[\theta M]^2) + \frac{1}{M}\,2\,\text{Kcub1}\,(\text{Cos}[\theta M]^4 + \text{Sin}[\theta M]^4\,(\text{Cos}[\phi M]^4 + \text{Sin}[\phi M]^4) - 3\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2\,(1 + \text{Cos}[\phi M]^4 + \text{Sin}[\phi M]^4)) + \frac{1}{M}\,2\,\text{Kcub2}\,(\text{Sin}[\theta M]^2\,\text{Cos}[\phi M]^2\,\text{Sin}[\phi M]^2\,(6\,\text{Cos}[\theta M]^4 + \text{Sin}[\theta M]^4 - 11\,\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2))\Big)$$

$$\Big(\text{Hx Sin}[\theta M]\,\text{Cos}[\phi M] + \text{Hy Sin}[\theta M]\,\text{Sin}[\phi M] + \text{Hz Cos}[\theta M] - M\left(\text{Nz} + \frac{\text{Kz}}{M^2}\right)\text{Cos}[\theta M]^2 + \frac{1}{M}\,2\,\text{Kcub1}\,(\text{Cos}[\theta M]^4 + \text{Sin}[\theta M]^4\,(\text{Cos}[\phi M]^4 + \text{Sin}[\phi M]^4) - 6\,\text{Sin}[\theta M]^2\,\text{Cos}[\phi M]^2\,\text{Sin}[\phi M]^2) + \frac{1}{M}\,2\,\text{Kcub2}\,(\text{Cos}[\theta M]^2\,\text{Sin}[\theta M]^2\,(\text{Cos}[\phi M]^4 + \text{Sin}[\phi M]^4 - (4 + 3\,\text{Sin}[\theta M]^2)\,\text{Cos}[\phi M]^2\,\text{Sin}[\phi M]^2))\Big) -$$

$$\Big(3\,\frac{1}{M}\,2\,\text{Kcub1}\,(\text{Cos}[\theta M]\,\text{Sin}[\theta M]^2\,\text{Cos}[\phi M]\,\text{Sin}[\phi M]\,(\text{Cos}[\phi M]^2 - \text{Sin}[\phi M]^2)) + \frac{1}{M}\,2\,\text{Kcub2}\,(\text{Cos}[\theta M]\,\text{Sin}[\theta M]^2\,(3\,\text{Cos}[\theta M]^2 - 2\,\text{Sin}[\theta M]^2)\,\text{Cos}[\phi M]\,\text{Sin}[\phi M]\,(\text{Cos}[\phi M]^2 - \text{Sin}[\phi M]^2))\Big)^2$$

For $\phi M = 0$: $Cos[\phi M] = 1$, $Sin[\phi M] = 0$

$$\left(\frac{\omega}{\gamma}\right)^2 =$$
$$\left(Hx\,Sin[\theta M] + Hz\,Cos[\theta M] + \left(-M\left(Nz + \frac{Kz}{M^2}\right)\right)(Cos[\theta M]^2 - Sin[\theta M]^2) + \frac{2\,Kcub1}{M}(Cos[\theta M]^4 + Sin[\theta M]^4\,(1) - 3\,Cos[\theta M]^2\,Sin[\theta M]^2\,(2))\right)$$
$$\left(Hx\,Sin[\theta M] + Hz\,Cos[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)Cos[\theta M]^2 + \frac{2\,Kcub1}{M}(Cos[\theta M]^4 + Sin[\theta M]^4) + \frac{2\,Kcub2}{M}(Cos[\theta M]^2\,Sin[\theta M]^2)\right)$$

For $\phi M = \frac{\pi}{2}$: $Cos[\phi M] = 0$, $Sin[\phi M] = 1$

We should get the same result

$$\left(\frac{\omega}{\gamma}\right)^2 =$$
$$\left(Hy\,Sin[\theta M] + Hz\,Cos[\theta M] + \left(-M\left(Nz + \frac{Kz}{M^2}\right)\right)(Cos[\theta M]^2 - Sin[\theta M]^2) + \frac{2\,Kcub1}{M}(Cos[\theta M]^4 + Sin[\theta M]^4\,(1) - 3\,Cos[\theta M]^2\,Sin[\theta M]^2\,(2))\right)$$
$$\left(Hy\,Sin[\theta M] + Hz\,Cos[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)Cos[\theta M]^2 + \frac{2\,Kcub1}{M}(Cos[\theta M]^4 + Sin[\theta M]^4\,(-1)) + \frac{2\,Kcub2}{M}(Cos[\theta M]^2\,Sin[\theta M]^2)\right)$$

We do $\rightarrow$ In - plane symmetric

For Magnetization In - *plane* : $Cos[\theta M] = 0$, $Sin[\theta M] = 1$

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(Hy + \left(-M\left(Nz + \frac{Kz}{M^2}\right)\right)(-1) + \frac{2\,Kcub1}{M}\right)\left(Hy + \frac{2\,Kcub1}{M}(-1)\right)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(Hy + \left(M\left(Nz + \frac{Kz}{M^2}\right)\right) + \frac{2\,Kcub1}{M}\right)\left(Hy - \frac{2\,Kcub1}{M}\right)$$

Ignoring fourth and sixth order terms :

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(Hy + \left(M\left(Nz + \frac{Kz}{M^2}\right)\right)\right)(Hy)$$

Ignoring *VCMA* :

$$\left(\frac{\omega}{\gamma}\right)^2 = (Hy + M\,Nz)\,(Hy)$$

$$\omega = \gamma\sqrt{(Hy + M\,Nz)\,(Hy)}$$

We get Kittel's result.

Backtracking to get angular dependence ignoring sixth order terms

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(Hy\,\mathrm{Sin}[\theta M] + Hz\,\mathrm{Cos}[\theta M] + \left(-M\left(Nz + \frac{Kz}{M^2}\right)\right)(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2) + \frac{2\,Kcub1}{M}(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right) \left(Hy\,\mathrm{Sin}[\theta M] + Hz\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,Kcub1}{M}(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4)\right)$$

$Hx = H\,\mathrm{Cos}[\phi H]\,\mathrm{Sin}[\theta H]$, $Hy = H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]$,
$Hz = H\,\mathrm{Cos}[\theta H]$

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2) + \frac{1}{M}\,2\,Kcub1\,(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right) \left(H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{1}{M}\,2\,Kcub1\,(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4)\right)$$

$\phi H = \frac{\pi}{2}$: $\mathrm{Cos}[\phi H] = 0$, $\mathrm{Sin}[\phi H] = 1$

$$\left(\frac{\omega}{\gamma}\right)^2 =$$

$$\left(H\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\left(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2\right) + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$
$$\left(H\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4\right)\right)$$

$$\mathrm{Simplify}\Big[\left(H\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\left(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2\right) + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$
$$\left(H\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4\right)\right)\Big]$$

$$\left(H\,\mathrm{Cos}[\theta H - \theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\left(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2\right) + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$
$$\left(H\,\mathrm{Cos}[\theta H - \theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4\right)\right)$$

$Kz = K1$, $Kcub1 = K2$

$$\left(H\,\mathrm{Cos}[\theta H - \theta M] - M\left(Nz + \frac{K1}{M^2}\right)\left(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2\right) + \frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$
$$\left(H\,\mathrm{Cos}[\theta H - \theta M] - M\left(Nz + \frac{K1}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(\mathrm{Cos}[\theta M]^2-\mathrm{Sin}[\theta M]^2\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4+\mathrm{Sin}[\theta M]^4-6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4-\mathrm{Sin}[\theta M]^4\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(\mathrm{Cos}[\theta M]^2-\left(1-\mathrm{Cos}[\theta M]^2\right)\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4+\left(2-\mathrm{Cos}[2\,\theta M]+\frac{1}{8}\left(\mathrm{Cos}[4\,\theta M]+4\,\mathrm{Cos}[2\,\theta M]+3\right)\right)-6\,\mathrm{Cos}[\theta M]^2\left(1-\mathrm{Cos}[\theta M]^2\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4-\left(2-\mathrm{Cos}[2\,\theta M]+\frac{1}{8}\left(\mathrm{Cos}[4\,\theta M]+4\,\mathrm{Cos}[2\,\theta M]+3\right)\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(\mathrm{Cos}[\theta M]^2-1+\mathrm{Cos}[\theta M]^2\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4+2-\mathrm{Cos}[2\,\theta M]+\frac{1}{8}\left(\mathrm{Cos}[4\,\theta M]+4\,\mathrm{Cos}[2\,\theta M]+3\right)-6\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4-\left(2-\mathrm{Cos}[2\,\theta M]+\frac{1}{8}\mathrm{Cos}[4\,\theta M]+\frac{1}{2}\mathrm{Cos}[2\,\theta M]+\frac{3}{8}\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(2\,\mathrm{Cos}[\theta M]^2-1\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4+2+\frac{1}{8}\mathrm{Cos}[4\,\theta M]-\frac{1}{2}\mathrm{Cos}[2\,\theta M]+\frac{3}{8}-6\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4-\left(2-\mathrm{Cos}[2\,\theta M]+\frac{1}{8}\mathrm{Cos}[4\,\theta M]+\frac{1}{2}\mathrm{Cos}[2\,\theta M]+\frac{3}{8}\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(2\left(\frac{\mathrm{Cos}[2\,\theta]+1}{2}\right)-1\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4+2+\frac{1}{8}\mathrm{Cos}[4\,\theta M]-\frac{1}{2}\mathrm{Cos}[2\,\theta M]+\frac{3}{8}-6\left(\left(\frac{\mathrm{Cos}[2\,\theta]+1}{2}\right)-\left(\frac{1}{8}(\mathrm{Cos}[4\,\theta]+4\,\mathrm{Cos}[2\,\theta]+3)\right)\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(\frac{\mathrm{Cos}[2\,\theta]+1}{2}\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4-\left(2-\mathrm{Cos}[2\,\theta M]+\frac{1}{8}\mathrm{Cos}[4\,\theta M]+\frac{1}{2}\mathrm{Cos}[2\,\theta M]+\frac{3}{8}\right)\right)\right)$$

Simplify[

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(2\left(\frac{\mathrm{Cos}[2\,\theta M]+1}{2}\right)-1\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4+2+\frac{1}{8}\mathrm{Cos}[4\,\theta M]-\frac{1}{2}\mathrm{Cos}[2\,\theta M]+\frac{3}{8}-6\left(\left(\frac{\mathrm{Cos}[2\,\theta M]+1}{2}\right)-\left(\frac{1}{8}(\mathrm{Cos}[4\,\theta M]+4\,\mathrm{Cos}[2\,\theta M]+3)\right)\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\left(\frac{\mathrm{Cos}[2\,\theta M]+1}{2}\right)+\frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4-\left(2-\mathrm{Cos}[2\,\theta M]+\frac{1}{8}\mathrm{Cos}[4\,\theta M]+\frac{1}{2}\mathrm{Cos}[2\,\theta M]+\frac{3}{8}\right)\right)\right)\Bigg]$$

$$\frac{1}{2\,M^2}\left(K1+8\,K2+M^2\,Nz-2\,H\,M\,\mathrm{Cos}[\theta H-\theta M]+(K1-4\,K2+M^2\,Nz)\,\mathrm{Cos}[2\,\theta M]\right)\left(-H\,M\,\mathrm{Cos}[\theta H-\theta M]+(K1+M^2\,Nz)\,\mathrm{Cos}[2\,\theta M]-2\,K2\,(2+\mathrm{Cos}[4\,\theta M])\right)$$

$$\frac{1}{2\,M^2}\left(-2\,H\,M\,\mathrm{Cos}[\theta H-\theta M]+K1+8\,K2+M^2\,Nz+(K1-4\,K2+M^2\,Nz)\,\mathrm{Cos}[2\,\theta M]\right)\left(-H\,M\,\mathrm{Cos}[\theta H-\theta M]+(K1+M^2\,Nz)\,\mathrm{Cos}[2\,\theta M]-2\,K2\,(2+\mathrm{Cos}[4\,\theta M])\right)$$

Thi does not seem to agree with Okada. However it as a lot of similar terms.

$\mathrm{Cos}[2\,\theta M]=2\,\mathrm{Cos}[\theta M]^2-1$

$\mathrm{Cos}[4\,\theta M]=2\,\mathrm{Cos}[2\,\theta M]^2-1$

$\mathrm{Cos}[4\,\theta M]=2\,(2\,\mathrm{Cos}[\theta M]^2-1)^2-1$

$\mathrm{Cos}[4\,\theta M]=2\,(4\,\mathrm{Cos}[\theta M]^4-4\,\mathrm{Cos}[\theta M]^2+1)-1$

$\mathrm{Cos}[4\,\theta M]=8\,\mathrm{Cos}[\theta M]^4-8\,\mathrm{Cos}[\theta M]^2+1$

$\mathrm{Cos}[\theta M]^2=\dfrac{\mathrm{Cos}[2\,\theta M]+1}{2}$

$$\mathrm{Cos}[\theta M]^4 = \frac{1}{8}\left(\mathrm{Cos}[4\,\theta M] + 8\,\mathrm{Cos}[\theta M]^2 - 1\right)$$

$$\mathrm{Cos}[\theta M]^4 = \frac{1}{8}\left(\mathrm{Cos}[4\,\theta M] + 8\left(\frac{\mathrm{Cos}[2\,\theta M] + 1}{2}\right) - 1\right)$$

$$\mathrm{Cos}[\theta M]^4 = \frac{1}{8}\left(\mathrm{Cos}[4\,\theta M] + 4\,\mathrm{Cos}[2\,\theta M] + 3\right)$$

$$\mathrm{Sin}[\theta M]^4 = (\mathrm{Sin}[\theta M]^2)^2 = (1 - \mathrm{Cos}[\theta M]^2)^2 = 1 - 2\,\mathrm{Cos}[\theta M]^2 + \mathrm{Cos}[\theta M]^4$$

$$\mathrm{Sin}[\theta M]^4 = 1 - 2\left(\frac{\mathrm{Cos}[2\,\theta M] + 1}{2}\right) + \frac{1}{8}\left(\mathrm{Cos}[4\,\theta M] + 4\,\mathrm{Cos}[2\,\theta M] + 3\right)$$

$$\mathrm{Sin}[\theta M]^4 = \left(2 - \mathrm{Cos}[2\,\theta M] + \frac{1}{8}\left(\mathrm{Cos}[4\,\theta M] + 4\,\mathrm{Cos}[2\,\theta M] + 3\right)\right)$$

Baselgia with sixth order terms angular dependent

$$\left(\frac{\omega}{\gamma}\right)^2 =$$
$$\left(Hy\,\mathrm{Sin}[\theta M] + Hz\,\mathrm{Cos}[\theta M] + \left(-M\left(Nz + \frac{Kz}{M^2}\right)\right)\left(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2\right) + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4\,(1) - 3\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\,(2)\right)\right)$$
$$\left(Hy\,\mathrm{Sin}[\theta M] + Hz\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{Kz}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,Kcub1}{M}\left(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4\,(-1)\right) + \frac{2\,Kcub2}{M}\left(\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$

$$\left(\frac{\omega}{\gamma}\right)^2 =$$
$$\left(Hy\,\mathrm{Sin}[\theta M] + Hz\,\mathrm{Cos}[\theta M] + \left(-M\left(Nz + \frac{K1}{M^2}\right)\right)\left(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2\right) + \frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$
$$\left(Hy\,\mathrm{Sin}[\theta M] + Hz\,\mathrm{Cos}[\theta M] - M\left(Nz + \frac{K1}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,K2}{M}\left(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4\right) + \frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$

$Hx = H\,\mathrm{Cos}[\phi H]\,\mathrm{Sin}[\theta H]$, $Hy = H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]$,
$Hz = H\,\mathrm{Cos}[\theta H]$

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] + \left(-M\left(Nz+\frac{K1}{M^2}\right)\right)(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2) + \frac{2\,K2}{M}(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right) \left(H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz+\frac{K1}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,K2}{M}(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4) + \frac{2\,K3}{M}(\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] + \left(-M\left(Nz+\frac{K1}{M^2}\right)\right)(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2) + \frac{2\,K2}{M}(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right) \left(H\,\mathrm{Sin}[\phi H]\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz+\frac{K1}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,K2}{M}(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4) + \frac{2\,K3}{M}(\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(H\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] + \left(-M\left(Nz+\frac{K1}{M^2}\right)\right)(\mathrm{Cos}[\theta M]^2 - \mathrm{Sin}[\theta M]^2) + \frac{2\,K2}{M}(\mathrm{Cos}[\theta M]^4 + \mathrm{Sin}[\theta M]^4 - 6\,\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right) \left(H\,\mathrm{Sin}[\theta H]\,\mathrm{Sin}[\theta M] + H\,\mathrm{Cos}[\theta H]\,\mathrm{Cos}[\theta M] - M\left(Nz+\frac{K1}{M^2}\right)\mathrm{Cos}[\theta M]^2 + \frac{2\,K2}{M}(\mathrm{Cos}[\theta M]^4 - \mathrm{Sin}[\theta M]^4) + \frac{2\,K3}{M}(\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2)\right)$$

```
Simplify[
 (H Sin[θH] Sin[θM] + H Cos[θH] Cos[θM] +
    (-M (Nz + K1/M^2)) (Cos[θM]^2 - Sin[θM]^2) +
    (2 K2)/M (Cos[θM]^4 + Sin[θM]^4 - 6 Cos[θM]^2 Sin[θM]^2))
  (H Sin[θH] Sin[θM] + H Cos[θH] Cos[θM] - M (Nz + K1/M^2) Cos[θM]^2 +
    (2 K2)/M (Cos[θM]^4 - Sin[θM]^4) + (2 K3)/M (Cos[θM]^2 Sin[θM]^2))]

-1/(4 M^2) (H M Cos[θH - θM] - (K1 + M^2 Nz) Cos[2 θM] + 2 K2 Cos[4 θM])
  (2 K1 - K3 + 2 M^2 Nz - 4 H M Cos[θH - θM] +
    2 (K1 - 4 K2 + M^2 Nz) Cos[2 θM] + K3 Cos[4 θM])

1/(4 M^2) (H M Cos[θH - θM] - (K1 + M^2 Nz) Cos[2 θM] + 2 K2 Cos[4 θM])
 (-2 K1 + K3 - 2 M^2 Nz + 4 H M Cos[θH - θM] - 2 (K1 - 4 K2 + M^2 Nz) Cos[2 θM] -
   K3 Cos[4 θM])
```

Manual Simplification

```
(H Cos[θH - θM] - M (Nz + K1/M^2) (Cos[θM]^2 - Sin[θM]^2) +
   (2 K2)/M (Cos[θM]^4 + Sin[θM]^4 - 6 Cos[θM]^2 Sin[θM]^2))
 (H Cos[θH - θM] - M (Nz + K1/M^2) Cos[θM]^2 + (2 K2)/M (Cos[θM]^4 - Sin[θM]^4) +
   (2 K3)/M (Cos[θM]^2 Sin[θM]^2))

(H Cos[θH - θM] - M (Nz + K1/M^2) Cos[2 θM] + (2 K2)/M Cos[4 θM])
 (H Cos[θH - θM] - M (Nz + K1/M^2) Cos[θM]^2 + (2 K2)/M Cos[2 θM] +
   (2 K3)/M (Cos[θM]^2 Sin[θM]^2))
```

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\mathrm{Cos}[2\,\theta M]+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\mathrm{Cos}[2\,\theta M]+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2\,\left(1-\mathrm{Cos}[\theta M]^2\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\mathrm{Cos}[2\,\theta M]+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\left(2\,\mathrm{Cos}[\theta M]^2-1\right)+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}+\frac{4\,K2}{M}\right)\mathrm{Cos}[\theta M]^2-\frac{2\,K2}{M}+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\mathrm{Cos}[2\,\theta M]+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2\,\mathrm{Sin}[\theta M]^2\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\mathrm{Cos}[2\,\theta M]+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2\left(1-\mathrm{Cos}[\theta M]^2\right)\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\mathrm{Cos}[2\,\theta M]+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[\theta M]^2+\frac{2\,K2}{M}\left(2\,\mathrm{Cos}[\theta M]^2-1\right)+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}\right)\mathrm{Cos}[2\,\theta M]+\frac{2\,K2}{M}\mathrm{Cos}[4\,\theta M]\right)$$

$$\left(H\,\mathrm{Cos}[\theta H-\theta M]-\left(M\,Nz+\frac{K1}{M}+\frac{4\,K2}{M}\right)\mathrm{Cos}[\theta M]^2-\frac{2\,K2}{M}+\frac{2\,K3}{M}\left(\mathrm{Cos}[\theta M]^2-\mathrm{Cos}[\theta M]^4\right)\right)$$

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